

On the Translative Packing Densities of Tetrahedra and Cubooctahedra

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Abstract. In 1900, as a part of his 18th problem, Hilbert asked the question to determine the density of the densest tetrahedron packings. However, up to now no mathematician knows the density $\delta^t(T)$ of the densest translative tetrahedron packings and the density $\delta^c(T)$ of the densest congruent tetrahedron packings. In this paper, by introducing a particular local method, we obtain the upper bound in

$$0.3673469 \cdots \leq \delta^t(T) \leq 0.3840610 \cdots,$$

where the lower bound was achieved by Groemer in 1962 which corrected a mistake of Minkowski. For the density $\delta^t(C)$ of the densest translative cubooctahedron packings, we obtain the upper bound in

$$0.9183673 \cdots \leq \delta^t(C) \leq 0.9601527 \cdots.$$

1. Introduction

In the extended version of his talk presented at the ICM 1900 in Paris, Hilbert [17] proposed 23 unsolved mathematical problems. At the end of his 18th problem, he asked “How can one arrange most densely in space an infinite number of equal solids of given form, e.g., spheres with given radii or regular tetrahedra with given edges (or in prescribed position), that is, how can one so fit them together that the ratio of the filled to the unfilled space may be as great as possible?” The spherical case, known as Kepler’s conjecture, has a much longer and more complicated history (see Hales [15] and [16]).

Let K denote a convex body in the three-dimensional Euclidean space \mathbb{E}^3 , with boundary $\partial(K)$ and interior $\text{int}(K)$. In particular, let T , O and C denote a regular tetrahedron, a regular octahedron and a regular cubooctahedron of unit edge, respectively. Let $\delta^c(K)$, $\delta^t(K)$ and $\delta^l(K)$ denote the densities of the densest congruent packings, the densest translative packings and the densest lattice packings of K , respectively. For basic results and open problems about these densities we refer to [2], [7], [8], [12], [13] and [23].

It follows by their definitions that

$$\delta^l(K) \leq \delta^t(K) \leq \delta^c(K) \leq 1 \quad (1.1)$$

holds for every convex body K . Let $\sigma(\mathbf{x})$ denote a nonsingular affine linear transformation from \mathbb{E}^3 to \mathbb{E}^3 . It is known (easy to verify) that both

$$\delta^l(\sigma(K)) = \delta^l(K)$$

and

$$\delta^t(\sigma(K)) = \delta^t(K)$$

hold for every K and any σ . However, for some objects K and suitable corresponding σ , we have

$$\delta^c(\sigma(K)) \neq \delta^c(K).$$

For example, when $\sigma(T)$ is the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$ and $(1, 1, 1)$, we have

$$\delta^c(T) < 1 \quad \text{and} \quad \delta^c(\sigma(T)) = 1.$$

When K is centrally symmetric and centered at the origin, in 1904 Minkowski [21] discovered the following criterion for its densest lattice packings: *If $K + \Lambda$ is a lattice packing of maximal density, then Λ has a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ such that either*

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_1 - \mathbf{a}_2, \mathbf{a}_2 - \mathbf{a}_3, \mathbf{a}_3 - \mathbf{a}_1\} \subset \partial(2K)$$

or

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_2 + \mathbf{a}_3, \mathbf{a}_3 + \mathbf{a}_1\} \subset \partial(2K).$$

As an application, he determined the density of the densest lattice packings of an octahedron O . In other words, he proved

$$\delta^l(O) = \frac{18}{19}. \quad (1.2)$$

Let K be a convex set and define

$$D(K) = \{\mathbf{x} - \mathbf{y} : \mathbf{x}, \mathbf{y} \in K\}.$$

Usually, we call $D(K)$ the difference set of K . Clearly $D(K)$ is a centrally symmetric convex set centered at the origin \mathbf{o} . It was discovered by Minkowski [21] that

$$(K + \mathbf{x}) \cap (K + \mathbf{y}) \neq \emptyset$$

if and only if

$$\left(\frac{1}{2}D(K) + \mathbf{x}\right) \cap \left(\frac{1}{2}D(K) + \mathbf{y}\right) \neq \emptyset.$$

Therefore, for a discrete set X in \mathbb{E}^3 , $K + X$ is a packing if and only if $\frac{1}{2}D(K) + X$ is a packing. Consequently, he proved

$$\delta^t(K) = \frac{2^3 v(K)}{v(D(K))} \cdot \delta^t(D(K)) \quad (1.3)$$

and

$$\delta^l(K) = \frac{2^3 v(K)}{v(D(K))} \cdot \delta^l(D(K)). \quad (1.4)$$

On page 312 of [21], Minkowski wrote “If K is a tetrahedron, then $\frac{1}{2}D(K)$ is an octahedron with faces parallel to the faces of the tetrahedron.” By routine computations, one can get $v(T) = \sqrt{2}/12$ and $v(O) = \sqrt{2}/3$. Then, by (1.2) and (1.4) Minkowski [21] made a conclusion that

$$\delta^l(T) = \frac{9}{38}. \quad (1.5)$$

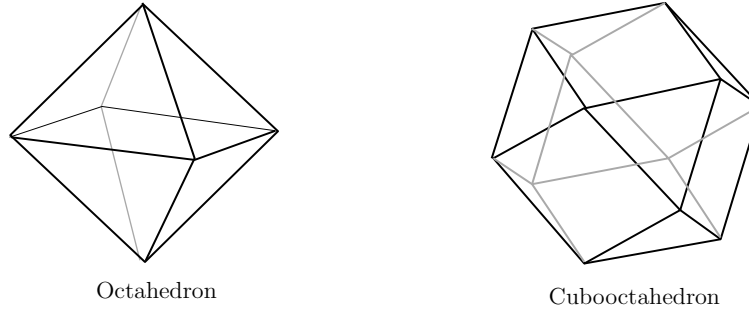


Figure 1

Minkowski's idea was brilliant. Unfortunately, he made a mistake, which was discovered by Groemer [11] in 1962. The difference set of a regular tetrahedron is not an octahedron, but a cubooctahedron. As it is shown in Figure 1, a cubooctahedron is very different from an octahedron. In fact, it was already known to Estermann [6] and Süss [25] in 1928 that

$$\frac{v(D(T))}{v(T)} = \frac{v(C)}{v(T)} = 20. \quad (1.6)$$

In 1970, Hoylman [18] applied Minkowski's criterion to a cubooctahedron C . By considering 38 cases with respect to the possible positions of the three vectors of the bases, he proved

$$\delta^l(C) = \frac{45}{49}, \quad (1.7)$$

$$\delta^l(T) = \frac{18}{49}, \quad (1.8)$$

and the optimal lattice is unique up to certain equivalence. In the densest lattice tetrahedron packing each tetrahedron touches 14 others. However, according to Zong [29], the density of the lattice tetrahedron packing of maximal kissing number 18 is only $\frac{1}{3}$.

Based on Minkowski's work, in 2000 Betke and Henk [1] developed an algorithm by which one can determine the density of the densest lattice packing of an arbitrary three-dimensional polytope. As an example of application, Hoylman's result was verified.

In 2006, Conway and Torquato [5] made a breakthrough in constructing dense congruent tetrahedron packings. Their idea is simple but very efficient. First, pack twenty regular tetrahedra into an icosahedron (see Figure 2). The fraction of the icosahedral volume occupied by the tetrahedra can be $0.8567627\dots$

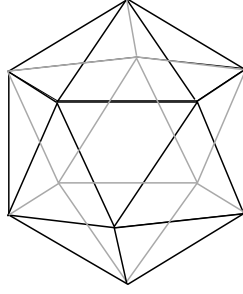


Figure 2

Then, construct a lattice icosahedron packing with maximum density. According to Betke and Henk [1] it is $0.8363574\dots$. Thus, we obtain a congruent tetrahedron packing of density

$$0.8363574 \times 0.8567627 \approx 0.716559.$$

In other words, we have

$$\delta^c(T) \geq 0.716559\dots \quad (1.9)$$

It was conjectured by S. Ulam (see page 135 of [9]) that the maximal density $0.74048\dots$ for packing congruent spheres is smaller than that for any other convex body. Of course, it makes sense to consider regular tetrahedron as a candidate of counterexample, as Conway and Torquato [5] did. In 2008, by constructing a cluster of eighteen congruent regular tetrahedra and a suitable lattice packing of the cluster, Conway and Torquato's lower bound (1.9) was improved by Chen [3] into

$$\delta^c(T) \geq 0.778615\dots,$$

which turns to support Ulam's conjecture.

Packings of regular tetrahedra may provide useful models in Material Sciences, Information Theory and etc. Therefore, recently it becomes an active research topic involving both mathematicians and scientists in other fields. Chen's lower bound was further improved by [26], [27], [14], [19], [4], [28] and etc. So far the best known lower bound is

$$\delta^c(T) \geq 0.856347\dots$$

According to Senechal [24], Aristotle believed that congruent regular tetrahedra can tile the space, which implies $\delta^c(T) = 1$. This error was discovered by J. Müller in the 15th century. However, the first nontrivial upper bound for $\delta^c(T)$ was achieved only in 2011. It was proved by Gravel, Elser and Kallus [10] that

$$\delta^c(T) \leq 1 - 2.6 \times 10^{-25}.$$

Up to now no reasonable conjecture on the exact value of $\delta^c(T)$ is known. For more information on tetrahedron packings we refer to Lagarias and Zong [20].

Perhaps, to determine the value of $\delta^t(T)$ is not as challenging as $\delta^c(T)$. However, since it is invariant under nonsingular linear transformations, $\delta^t(T)$ seems more important. By (1.1), (1.3), (1.7) and (1.8) one can deduce

$$\frac{45}{49} \leq \delta^t(C) \leq 1$$

and

$$\frac{18}{49} \leq \delta^t(T) \leq \frac{2}{5}.$$

These facts support the conjecture that

$$\delta^t(C) = \frac{45}{49} \quad \text{and} \quad \delta^t(T) = \frac{18}{49}.$$

In this paper, a particular local method to estimate $\delta^t(K)$ is presented. As a application to $\delta^t(C)$ and $\delta^t(T)$, we prove the following result:

The Main Theorem.

$$\delta^t(C) \leq \frac{90\sqrt{10}}{95\sqrt{10}-4} \quad \text{and} \quad \delta^t(T) \leq \frac{36\sqrt{10}}{95\sqrt{10}-4}.$$

Combined with (1.7) and (1.8) we get

Corollary 1.1.

$$0.9183673 \cdots \leq \delta^t(C) \leq 0.9601527 \cdots$$

and

$$0.3673469 \cdots \leq \delta^t(T) \leq 0.3840610 \cdots$$

Remark 1.1. *To read this paper, several cubooctahedron models can be helpful.*

2. The Key Strategy and Basic Terminology

For convenience, let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of \mathbb{E}^3 , and write

$$B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$$

and

$$W = \{(x, y, z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}.$$

According to John's theorem (see page 13 of [13]), for every three-dimensional convex body K there is a nonsingular affine linear transformation σ from \mathbb{E}^3 to \mathbb{E}^3 such that

$$\frac{1}{3}B \subseteq \sigma(K) \subseteq B$$

and therefore

$$\frac{\sqrt{3}}{9}W \subseteq \sigma(K) \subseteq W.$$

On the other hand, it is well-known that

$$\delta^t(\sigma(K)) = \delta^t(K)$$

holds for every convex body K and any nonsingular affine linear transformation σ . Thus, to study $\delta^t(K)$, it is sufficient to work on convex bodies K satisfying

$$\frac{\sqrt{3}}{9}W \subseteq K \subseteq W. \quad (2.1)$$

In particular, we write

$$C = \{(x, y, z) : \max\{|x|, |y|, |z|\} \leq 1, |x| + |y| + |z| \leq 2\}.$$

In fact, the cubooctahedron C can be obtained from the cube W by cutting off eight orthogonal unit tetrahedra. Thus we have

$$v(C) = 8 \left(1 - \frac{1}{6}\right) = \frac{20}{3}. \quad (2.2)$$

Take K to be a convex body satisfying (2.1) and assume that $X = \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots\}$ is a discrete set of points such that $K + X$ is a translative packing in \mathbb{E}^3 and let \mathcal{X} denote the family of all such sets. For convenience we take \mathbf{x}_0 to be the origin \mathbf{o} of the space and denote the packing density of $K + X$ by $\delta(K, X)$. In other words,

$$\delta(K, X) = \limsup_{\ell \rightarrow \infty} \frac{n(\ell) \cdot v(K)}{v(\ell W)},$$

where $n(\ell) = |X \cap \ell W|$.

Let \mathbf{v} be a unit vector and let $s(K, X, \mathbf{v}, \mathbf{x})$ denote the set of points \mathbf{y} such that $\mathbf{y} = \mathbf{x} + \tau \mathbf{v}$ holds for some positive number τ and the whole open segment (\mathbf{x}, \mathbf{y}) belongs to $\mathbb{E}^3 \setminus \{K + X\}$. Then, we define $D(K, X, \mathbf{v}, \mathbf{x}_j)$ to be the closure of

$$\left(\bigcup_{\mathbf{x} \in \partial(K + \mathbf{x}_j)} s(K, X, \mathbf{v}, \mathbf{x}) \right) \cap (2W + \mathbf{x}_j).$$

In this paper we mainly deal with the cases $\mathbf{v} = \mathbf{e}_1, \mathbf{e}_2$ or \mathbf{e}_3 .

The cube $2W + \mathbf{x}_j$ here localizes the considered region, which will simplify the computation in next sections. Clearly $D(K, X, \mathbf{e}_i, \mathbf{x}_j)$ is a measurable set associated to $K + \mathbf{x}_j$ and, for fixed index i ,

$$\text{int}(D(K, X, \mathbf{e}_i, \mathbf{x}_j)) \cap \text{int}(D(K, X, \mathbf{e}_i, \mathbf{x}_k)) = \emptyset$$

holds for any pair of distinct indices j and k . Therefore the fraction

$$\frac{v(K)}{v(K) + v(D(K, X, \mathbf{e}_i, \mathbf{x}_j))}$$

defines a local packing density of $K + X$ at $K + \mathbf{x}_j$.

Example 2.1. When

$$\Lambda_1 = \left\{ \sum 2z_i \mathbf{e}_i : z_i \in \mathbb{Z} \right\},$$

the set $D(C, \Lambda_1, \mathbf{e}_1, \mathbf{o})$ is the union of eight tetrahedra, each of them is congruent to the one with vertices $(1, 1, 1)$, $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$. Therefore, for all $i = 1, 2, 3$ and $\mathbf{u}_j \in \Lambda_1$, we get

$$v(D(C, \Lambda_1, \mathbf{e}_i, \mathbf{u}_j)) = \frac{4}{3},$$

and by (2.2)

$$\delta(C, \Lambda_1) = \frac{v(C)}{v(C) + v(D(C, \Lambda_1, \mathbf{e}_1, \mathbf{o}))} = \frac{5}{6}.$$

Example 2.2. When $\mathbf{a}_1 = (2, -\frac{1}{3}, -\frac{1}{3})$, $\mathbf{a}_2 = (-\frac{1}{3}, 2, -\frac{1}{3})$, $\mathbf{a}_3 = (-\frac{1}{3}, -\frac{1}{3}, 2)$ and

$$\Lambda_2 = \left\{ \sum z_i \mathbf{a}_i : z_i \in \mathbb{Z} \right\}, \quad (2.3)$$

we get

$$D(C, \Lambda_2, \mathbf{e}_i, \mathbf{u}_j) = \frac{16}{27}$$

and

$$\delta(C, \Lambda_2) = \frac{v(C)}{v(C) + v(D(C, \Lambda_2, \mathbf{e}_1, \mathbf{o}))} = \frac{45}{49}.$$

We define

$$\bar{\mu}(K, X) = \min_{\mathbf{x}_j \in X} \frac{1}{3} \sum_{i=1,2,3} v(D(K, X, \mathbf{e}_i, \mathbf{x}_j)) \quad (2.4)$$

and

$$\mu(K, X) = \min_{\mathbf{x}_j \in X} \min_{i=1,2,3} v(D(K, X, \mathbf{e}_i, \mathbf{x}_j)). \quad (2.5)$$

Then we have

$$\mu(K, X) \leq \bar{\mu}(K, X) \quad (2.6)$$

and

$$\delta(K, X) \leq \frac{v(K)}{v(K) + \bar{\mu}(K, X)} \leq \frac{v(K)}{v(K) + \mu(K, X)}. \quad (2.7)$$

Furthermore, we define

$$\mu(K) = \min_{X \in \mathcal{X}} \mu(K, X) \quad (2.8)$$

and

$$\bar{\mu}(K) = \min_{X \in \mathcal{X}} \bar{\mu}(K, X). \quad (2.9)$$

By (2.6) and (2.7) we get

$$\mu(K) \leq \bar{\mu}(K) \quad (2.10)$$

and

$$\delta^t(K) \leq \frac{v(K)}{v(K) + \bar{\mu}(K)} \leq \frac{v(K)}{v(K) + \mu(K)}. \quad (2.11)$$

Thus, we have proved the following result, which is one of the keys for this paper.

Lemma 2.1.

$$\delta^t(K) \leq \frac{v(K)}{v(K) + \bar{\mu}(K)} \leq \frac{v(K)}{v(K) + \mu(K)}.$$

For many pairs $\{K, X\}$ we have identity

$$\mu(K, X) = \bar{\mu}(K, X).$$

However, this is not always true. For example, if we take $\mathbf{b}_1 = (2, 0, 0)$, $\mathbf{b}_2 = (1, \frac{3}{2}, \frac{3}{2})$, $\mathbf{b}_3 = (1, -\frac{3}{2}, \frac{3}{2})$ and

$$\Lambda_3 = \left\{ \sum z_i \mathbf{b}_i : z_i \in \mathbb{Z} \right\}, \quad (2.12)$$

then we have

$$\mu(C, \Lambda_3) = \frac{1}{3}$$

and

$$\bar{\mu}(C, \Lambda_3) = \frac{1}{3} \left(\frac{1}{3} + 2 + \frac{1}{4} + 2 + \frac{1}{4} \right) = \frac{29}{18}.$$

Lemma 2.2. *For any three-dimensional convex body K satisfying (2.1) there are two suitable finite discrete sets X_1 and X_2 satisfying*

$$\mu(K) = v(D(K, X_1, \mathbf{e}_1, \mathbf{o}))$$

and

$$\bar{\mu}(K) = \frac{1}{3} \sum_{i=1,2,3} v(D(K, X_2, \mathbf{e}_i, \mathbf{o})).$$

Proof. Suppose that Q_1, Q_2, Q_3, \dots is a sequence of discrete sets in \mathbb{E}^3 and $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \dots$ is a corresponding sequence of points such that $K + Q_i$ are packings in \mathbb{E}^3 , $\mathbf{q}_i \in Q_i$, and

$$\lim_{i \rightarrow \infty} v(D(K, Q_i, \mathbf{e}_1, \mathbf{q}_i)) = \mu(K). \quad (2.13)$$

By defining

$$Q'_i = \{\mathbf{q} - \mathbf{q}_i : \mathbf{q} \in Q_i\}$$

and

$$Q_i^* = Q'_i \cap 3W,$$

it follows from the definition of $D(K, X, \mathbf{e}_i, \mathbf{x}_j)$ that

$$D(K, Q_i^*, \mathbf{e}_1, \mathbf{o}) = D(K, Q'_i, \mathbf{e}_1, \mathbf{o}) = D(K, Q_i, \mathbf{e}_1, \mathbf{q}_i) - \mathbf{q}_i$$

and therefore by (2.13)

$$\begin{aligned} \lim_{i \rightarrow \infty} v(D(K, Q_i^*, \mathbf{e}_1, \mathbf{o})) &= \lim_{i \rightarrow \infty} v(D(K, Q'_i, \mathbf{e}_1, \mathbf{o})) \\ &= \lim_{i \rightarrow \infty} v(D(K, Q_i, \mathbf{e}_1, \mathbf{q}_i)) \\ &= \mu(K). \end{aligned} \quad (2.14)$$

Notice that

$$K + \mathbf{q}^* \subset 4W$$

whenever $\mathbf{q}^* \in Q_i^*$. Let $|X|$ denote the number of the points of X , by (2.1) we obtain

$$|Q_i^*| \leq \frac{v(4W)}{v(K)} = \frac{8^3 \cdot 9^3}{\sqrt{3}^3} \leq 71,832.$$

Therefore, by a selection process we can obtain a subsequence $Q_{i_1}^*, Q_{i_2}^*, Q_{i_3}^*, \dots$ of the sequence $Q_1^*, Q_2^*, Q_3^*, \dots$ and a certain discrete set X_1 such that

$$\lim_{j \rightarrow \infty} Q_{i_j}^* = X_1.$$

Clearly, $K + X_1$ is a packing in \mathbb{E}^3 and

$$v(D(K, X_1, \mathbf{e}_1, \mathbf{o})) = \lim_{j \rightarrow \infty} v(D(K, Q_{i_j}^*, \mathbf{e}_1, \mathbf{o})) = \mu(K).$$

The second part can be proved by a similar argument. \square

Remark 2.1. *Lemma 2.2 and Lemma 2.1 provide a mean to determine or estimate the values of $\mu(K)$, $\bar{\mu}(K)$ and therefore $\delta^t(K)$. In this paper we will consider the particular case $K = C$.*

Now we present two simple observations which will be frequently used in checking certain polytope is in the holes of $\{C + X\}$. Assume that $X = \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots\}$ is a discrete set such that $C + X$ is a cuboctahedron packing in \mathbb{E}^3 and let F be a certain set of points. For convenience, we use $\mathbf{x}_i \mathbf{x}_j \prec F$ to abbreviate the statement that $\mathbf{x}_i + \lambda(\mathbf{x}_j - \mathbf{x}_i) \in F$ holds for some positive number λ .

Lemma 2.3. *If $\mathbf{x}_i \mathbf{x}_j \prec F_i$, where F_i is a facet of $C + \mathbf{x}_i$, then the hyperplane generated by F_i separates $\text{int}(C) + \mathbf{x}_i$ and $\text{int}(C) + \mathbf{x}_j$.*

The fact is obvious. A proof is not necessary.

Lemma 2.4. *If F is a triangular facet of C , two interiorly disjoint translates of C can not simultaneously touch C at the interior of F .*

Proof. Without loss of generality, we assume that F is the facet with vertices $(1, 0, 1)$, $(1, 1, 0)$ and $(0, 1, 1)$. Let D denote the hexagon with vertices $(-1, 0, 1)$, $(0, -1, 1)$, $(1, -1, 0)$, $(1, 0, -1)$, $(0, 1, -1)$ and $(-1, 1, 0)$, and let $\rho(\mathbf{x}, \mathbf{y})$ denote the metric defined by D on any hyperplane which is parallel with D . Clearly F and D are parallel to each other. If two translates $C + \mathbf{x}_i$ and $C + \mathbf{x}_j$ can simultaneously touch C at the interior of F . Then, both $\frac{1}{2}\mathbf{x}_i$ and $\frac{1}{2}\mathbf{x}_j$ are interior points of F , and $\mathbf{x}_i \mathbf{x}_j$ is parallel with the hyperplane generated by D . Thus we get

$$\rho(\frac{1}{2}\mathbf{x}_i, \frac{1}{2}\mathbf{x}_j) < 1,$$

$$\rho(\mathbf{x}_i, \mathbf{x}_j) < 2$$

and therefore

$$(int(C) + \mathbf{x}_i) \cap (int(C) + \mathbf{x}_j) \neq \emptyset,$$

which contradicts the assumption of the lemma. Lemma 2.4 is proved. \square

Let us end this section by a conjecture and a corresponding remark.

Conjecture 2.1.

$$\overline{\mu}(C) = \frac{16}{27}.$$

Remark 2.2. *This conjecture implies both*

$$\delta^t(C) = \frac{45}{49} \quad \text{and} \quad \delta^t(T) = \frac{18}{49}.$$

3. Localization I, An Observation

Let us divide the set $R = \{(x, y, z) \in \partial(C) : x \geq 0\}$ into four parts

$$R_1 = \{(x, y, z) \in R : y \geq 0, z \geq 0\},$$

$$R_2 = \{(x, y, z) \in R : y \geq 0, z \leq 0\},$$

$$R_3 = \{(x, y, z) \in R : y \leq 0, z \leq 0\},$$

and

$$R_4 = \{(x, y, z) \in R : y \leq 0, z \geq 0\}.$$

Recall that $s(C, X, \mathbf{e}_i, \mathbf{x})$ is the longest open segment (\mathbf{x}, \mathbf{y}) such that $\mathbf{y} = \mathbf{x} + \tau \mathbf{e}_i$ holds with some $\tau > 0$ and

$$(\mathbf{x}, \mathbf{y}) \subset \mathbb{E}^3 \setminus \{C + X\}.$$

Then, for $i = 1, 2, 3$ and 4, we define $D_i(C, X)$ to be the closure of

$$\left(\bigcup_{\mathbf{x} \in R_i} s(C, X, \mathbf{e}_i, \mathbf{x}) \right) \cap 2W$$

and define

$$\mu_i(C) = \min_{X \in \mathcal{X}} v(D_i(C, X)). \quad (3.1)$$

Similar to Lemma 2.2, one can deduce the following fact:

Lemma 3.1. *For $i = 1, 2, 3$ and 4, there are suitable finite discrete sets X_i satisfying*

$$\mu_i(C) = v(D_i(C, X_i)).$$

It is easy to see that

$$D(C, X, \mathbf{e}_1, \mathbf{o}) = \bigcup_{i=1}^4 D_i(C, X) \quad (3.2)$$

and

$$int(D_i(C, X)) \cap int(D_j(C, X)) = \emptyset \quad (3.3)$$

holds for distinct i and j . On the other hand, by symmetry, we have

$$\mu_1(C) = \mu_2(C) = \mu_3(C) = \mu_4(C). \quad (3.4)$$

Therefore, by (2.8), (2.5), (3.1)-(3.4) we get

$$\begin{aligned} \mu(C) &= \min_{X \in \mathcal{X}} v(D(C, X, \mathbf{e}_1, \mathbf{o})) \\ &= \min_{X \in \mathcal{X}} \sum_{i=1}^4 v(D_i(C, X)) \\ &\geq \sum_{i=1}^4 \min_{X \in \mathcal{X}} v(D_i(C, X)) \\ &= 4 \cdot \mu_1(C). \end{aligned} \quad (3.5)$$

Thus, we have proved the following result.

Lemma 3.2.

$$\mu(C) \geq 4 \cdot \mu_1(C).$$

Based on Lemma 3.1, we assume that X is a discrete finite set of smallest cardinality such that $C + X$ is a packing in \mathbb{E}^3 and

$$\mu_1(C) = v(D_1(C, X)). \quad (3.6)$$

By computing the value of $v(D_1(C, \Lambda_3))$, where Λ_3 is defined by (2.12), we obtain

$$v(D_1(C, X)) \leq v(D_1(C, \Lambda_3)) = \frac{1}{6} \times \frac{1}{2}. \quad (3.7)$$

For convenience, we write $X' = X \setminus \{\mathbf{o}\}$ and enumerate the points of X' with non-decreasing x -coordinates. Then \mathbf{x}_1 has the smallest x -coordinate among the points in X' and

$$X' \subset \{(x, y, z) : 0 < x < 3, -1 < y < 2, -1 < z < 2\} \setminus \text{int}(2C). \quad (3.8)$$

If $\mathbf{x}_1 = (x_1, y_1, z_1) \notin \partial(2C)$ and ϵ is a positive number, we write $\mathbf{x}'_1 = (x_1 - \epsilon, y_1, z_1)$ and

$$X_1 = \mathbf{x}'_1 \cup (X \setminus \{\mathbf{x}_1\}).$$

When ϵ is small one can deduce that $C + X_1$ is a packing in \mathbb{E}^3 and

$$v(D_1(C, X_1)) < v(D_1(C, X)),$$

which contradicts the minimal assumption on $v(D_1(C, X))$. Thus we get

$$\mathbf{x}_1 \in \partial(2C).$$

In other words, $C + \mathbf{x}_1$ touches C at its boundary.

For convenience, we write

$$\begin{aligned} F_0 &= \{(x, y, z) : x = 1, |y| + |z| \leq 1\}, \\ F_1 &= \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0, x + y + z = 2\}, \\ F_2 &= \{(x, y, z) : x \geq 0, y \leq 0, z \geq 0, x - y + z = 2\}, \\ F_3 &= \{(x, y, z) : x \geq 0, y \leq 0, z \leq 0, x - y - z = 2\}, \\ F_4 &= \{(x, y, z) : x \geq 0, y \geq 0, z \leq 0, x + y - z = 2\} \end{aligned}$$

and $\mathbf{x}_0 = \mathbf{o}$. In fact, F_0 is a square facet of C , and F_1, F_2, F_3 and F_4 are four triangular facets surrounding F_0 . Recall that if $\mathbf{x}_0 \mathbf{x}_i \prec F_j$ holds for some $\mathbf{x}_i \in X'$, then the hyperplane generated by F_j separates $\text{int}(C)$ and $\text{int}(C) + \mathbf{x}_i$. Thus, by the minimal assumption on the cardinality of X , one can deduced that

$$\mathbf{x}_0 \mathbf{x}_i \prec F_0 \cup \text{int}(F_1) \cup \text{int}(F_2) \cup \text{int}(F_4) \quad (3.9)$$

holds for all points $\mathbf{x}_i \in X'$. In particular, X' has a point $\mathbf{x}' = (x', y', z')$ satisfying $\mathbf{x}_0 \mathbf{x}' \prec F_1$, $0 < x' < 2$, $0 < y' < 2$ and $0 < z' < 2$. Otherwise, the tetrahedron with vertices $(1, 1, 1)$, $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$ would be a subset of $D_1(C, X)$ and consequently

$$v(D_1(C, X)) \geq \frac{1}{6},$$

which contradicts to (3.7). Without loss of generality, we assume that \mathbf{x}' has the smallest x -coordinate among all points of this kind in X .

Let $\mathbf{x}_i = (x_i, y_i, z_i) \in X'$ be a point satisfying $\mathbf{x}_0 \mathbf{x}_i \prec F_4$ (or $\mathbf{x}_0 \mathbf{x}_i \prec F_2$), and let D denote the regular hexagon with vertices $(1, 0, 1)$, $(1, 1, 0)$, $(0, 1, -1)$, $(-1, 0, -1)$, $(-1, -1, 0)$ and $(0, -1, 1)$. If $x' + y' + z' > 4$, then $C + \mathbf{x}_i$ can not prevent $C + \mathbf{x}'$ from touching C along the $-\mathbf{e}_1$ direction. In fact, if $C + \mathbf{x}_i$ blocks $C + \mathbf{x}'$ from moving in the $-\mathbf{e}_1$ direction at $F_1 + \mathbf{x}_i$, then by (3.8) we get

$$\begin{cases} x_i + y_i - z_i \geq 4 \\ (x' - x_i) + (y' - y_i) + (z' - z_i) = 4 \end{cases}$$

and therefore

$$x' + y' = 4 + x_i + y_i + (z_i - z') \geq 8 + z_i + (z_i - z') \geq 4,$$

which contradicts the assumption $y' < 2$ and $z' < 2$. If $C + \mathbf{x}_i$ blocks $C + \mathbf{x}'$ from moving in the $-\mathbf{e}_1$ direction by $D + \mathbf{x}_i$, noticing that C is contained in the cylinder with base D and axis $(0, 1, 1)$, then we get $x' \geq 2$ which contradicts the assumption on \mathbf{x}' as well. Thus, defining $\mathbf{x}^* = (x' - \epsilon, y', z')$ and taking

$$X_2 = \mathbf{x}^* \cup (X \setminus \{\mathbf{x}'\}),$$

when ϵ is a suitable small positive number one can deduce that $C + X_2$ is a packing in \mathbb{E}^3 and

$$v(D_1(C, X_2)) < v(D_1(C, X)),$$

which contradicts the minimal assumption on $v(D_1(C, X))$. As a conclusion we have proved the following assertion.

Lemma 3.3. *If $C + X$ is a packing such that $\mu_1(C) = v(D_1(C, X))$, then there is a point $(x, y, z) \in X$ satisfying $0 < x < 2$, $0 < y < 2$, $0 < z < 2$ and $x + y + z = 4$. In other words, there is a translate $C + \mathbf{x}$ in $C + X$ which touches C at some interior point of F_1 .*

Now we prove the following result.

Theorem 3.1.

$$\mu_1(C) \geq \frac{1}{6} \times \frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}}.$$

Proof. Let T_0 denote the orthogonal tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, let F_1 denote the triangular facet of C with vertices $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$, and let X be a discrete set such that

$$\mu_1(C) = v(D_1(C, X)).$$

It follows by Lemma 3.3 that there is a $\mathbf{x}_1 = (x_1, y_1, z_1) \in X$ such that $C + \mathbf{x}_1$ touches C at the interior of F_1 . Then $C \cap (C + \mathbf{x}_1)$ is either a centrally symmetric hexagon or a parallelogram.

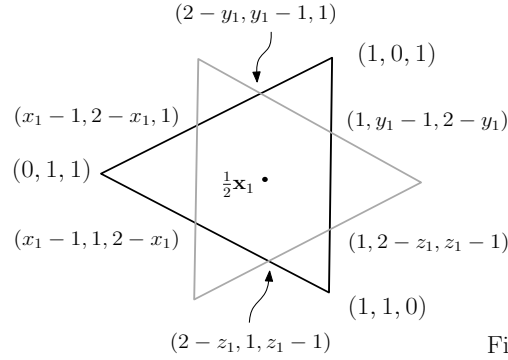


Figure 3

When $C \cap (C + \mathbf{x}_1)$ is an hexagon, its vertices are $(1, 2 - z_1, z_1 - 1)$, $(1, y_1 - 1, 2 - y_1)$, $(2 - y_1, y_1 - 1, 1)$, $(x_1 - 1, 2 - x_1, 1)$, $(x_1 - 1, 1, 2 - x_1)$ and $(2 - z_1, 1, z_1 - 1)$, as shown in Figure 3, where

$$x_1 + y_1 + z_1 = 4. \quad (3.10)$$

Let T_1 denote the tetrahedron with vertices $(x_1 - 1, 1, 1)$, $(0, 1, 1)$, $(x_1 - 1, 2 - x_1, 1)$ and $(x_1 - 1, 1, 2 - x_1)$, let T_2 denote the tetrahedron with vertices $(1, y_1 - 1, 1)$, $(2 - y_1, y_1 - 1, 1)$, $(1, 0, 1)$ and $(1, y_1 - 1, 2 - y_1)$, let T_3 denote the tetrahedron with vertices $(1, 1, z_1 - 1)$, $(2 - z_1, 1, z_1 - 1)$, $(1, 2 - z_1, z_1 - 1)$ and $(1, 1, 0)$, and let T_4 denote the tetrahedron with vertices $(1, y_1 - 1, z_1 - 1)$, $(x_1, y_1 - 1, z_1 - 1)$, $(1, 2 - z_1, z_1 - 1)$ and $(1, y_1 - 1, 2 - y_1)$. Clearly, all T_1 , T_2 , T_3 and T_4 are homothetic to T_0 with ratios $x_1 - 1$, $y_1 - 1$, $z_1 - 1$ and $x_1 - 1$, respectively, and

$$\text{int}(T_i) \cap \text{int}(T_j) = \emptyset$$

holds whenever $i \neq j$. By routine arguments based on Lemmas 2.3 and 2.4 we get

$$T_i \subseteq D_1(C, X)$$

for all $i = 1, 2, 3, 4$. Thus, by (3.10), we get

$$\begin{aligned} v(D_1(C, X)) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4) \\ &= \frac{1}{6} (2(x_1 - 1)^3 + (y_1 - 1)^3 + (z_1 - 1)^3) \\ &\geq \frac{1}{6} \left(2(x_1 - 1)^3 + 2 \left(1 - \frac{1}{2}x_1 \right)^3 \right) \\ &\geq \frac{1}{6} \times \frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}}, \end{aligned} \quad (3.11)$$

where the last equality holds if and only if $x_1 = 1 + \frac{1}{2\sqrt{2}+1}$.

When $C \cap (C + \mathbf{x}_1)$ is a parallelogram, by similar arguments, it can be proved that

$$v(D_1(C, X)) \geq \frac{1}{6} \times \frac{1}{4}. \quad (3.12)$$

As a conclusion of (3.11) and (3.12), noticing that

$$\frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}} \approx 0.1364549 \dots,$$

Theorem 3.1 is proved. □

Corollary 3.1.

$$\delta^t(C) \leq 0.9865382 \dots \quad \text{and} \quad \delta^t(T) \leq 0.3946153 \dots.$$

4. Localization II, A Detailed Computation

In this section we prove the following result.

Theorem 4.1.

$$\mu_1(C) = \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right).$$

Remark 4.1. We notice that

$$\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \approx 0.415009881 \dots.$$

For convenience, we write

$$\alpha = \frac{1}{3} \left(2 + 4 \cos \frac{\pi + \arctan \sqrt{63}}{3} \right) \approx 0.7223517 \dots$$

and

$$\beta = 1 + \sqrt[3]{\frac{1}{2}} \approx 1.7937005 \dots.$$

In fact, they are roots of the equations $3x^3 - 6x^2 + 2 = 0$ and $(x - 1)^3 - \frac{1}{2} = 0$, respectively.

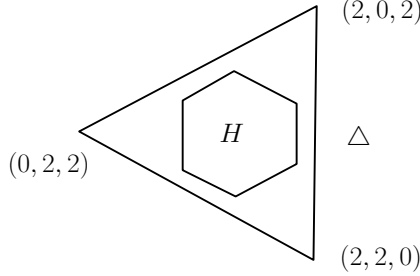


Figure 4

Let Δ denote the triangle with vertices $(0, 2, 2)$, $(2, 0, 2)$ and $(2, 2, 0)$, as illustrated in Figure 4, and write

$$H = \{(x, y, z) : \alpha \leq x \leq \beta, \alpha \leq y \leq \beta, \alpha \leq z \leq \beta, x + y + z = 4\}.$$

Lemma 4.1. If $\mathbf{x}_1 = (x_1, y_1, z_1) \in \Delta \setminus H$, we have

$$v(D_1(C, X)) \geq \frac{1}{6} \times \frac{1}{2}.$$

Proof. When $x_1 \geq \beta$, by Lemma 2.3 and Lemma 2.4 it can be shown that the tetrahedron T_1 with vertices $(x_1 - 1, 1, 1)$, $(0, 1, 1)$, $(x_1 - 1, 2 - x_1, 1)$ and $(x_1 - 1, 1, 2 - x_1)$ is a subset of $D_1(C, X)$. The tetrahedron T_1 is homothetic to the orthogonal tetrahedron T_0 defined in the proof of Theorem 3.1 with ratio $x_1 - 1$. Thus, we get

$$v(D_1(C, X)) \geq v(T_1) = \frac{1}{6}(x_1 - 1)^3 \geq \frac{1}{6} \left(\sqrt[3]{\frac{1}{2}} \right)^3 = \frac{1}{6} \times \frac{1}{2}. \quad (4.1)$$

The cases $y_1 \geq \beta$ and $z_1 \geq \beta$ can be treated by similar arguments.

When $x_1 \leq \alpha$, by Lemmas 2.3 and 2.4 it can be verified that both tetrahedra T_2 and T_3 are contained in $D_1(C, X)$, where T_2 has vertices $(1, y_1 - 1, 1)$, $(2 - y_1, y_1 - 1, 1)$, $(1, 0, 1)$ and $(1, y_1 - 1, 2 - y_1)$, T_3 has vertices

$(1, 1, z_1 - 1)$, $(2 - z_1, 1, z_1 - 1)$, $(1, 2 - z_1, z_1 - 1)$ and $(1, 1, 0)$. Clearly all T_2 , T_3 and $T_2 \cap T_3$ are homothetic to T_0 with ratios $y_1 - 1$, $z_1 - 1$ and $1 - x_1$, respectively. Thus, recalling the definition of α , we have

$$\begin{aligned}
 v(D_1(C, X)) &\geq v(T_2 \cup T_3) = v(T_2) + v(T_3) - v(T_2 \cap T_3) \\
 &= \frac{1}{6} ((y_1 - 1)^3 + (z_1 - 1)^3 - (1 - x_1)^3) \\
 &\geq \frac{1}{6} \left(2 \left(1 - \frac{1}{2} x_1 \right)^3 - (1 - x_1)^3 \right) \\
 &\geq \frac{1}{6} \left(2 \left(1 - \frac{1}{2} \alpha \right)^3 - (1 - \alpha)^3 \right) \\
 &= \frac{1}{6} \times \frac{1}{2}.
 \end{aligned} \tag{4.2}$$

The cases $y_1 \leq \alpha$ and $z_1 \leq \alpha$ can be proved by similar arguments.

As a conclusion of (4.1) and (4.2) the lemma is proved. \square

Assume that X is a discrete set of points such that $C + X$ is a packing in \mathbb{E}^3 satisfying

$$v(D_1(C, X)) = \mu_1(C).$$

Let Δ denote the triangle with vertices $(0, 2, 2)$, $(2, 0, 2)$ and $(2, 2, 0)$, let Δ^* denote the triangle with vertices $(2, 1, 1)$, $(1, 2, 1)$ and $(1, 1, 2)$, let Δ_1 denote the triangle with vertices $(2, 0, 2)$, $(2, 0.5, 1.5)$ and $(1, 1, 2)$, let Δ_2 denote the triangle with vertices $(2, 0.5, 1.5)$, $(2, 1, 1)$ and $(1, 1, 2)$, let Δ_3 denote the triangle with vertices $(2, 1, 1)$, $(1, 1, 2)$ and $(1, 1.5, 1.5)$, let Δ_4 denote the triangle with vertices $(1, 1.5, 1.5)$, $(1, 1, 2)$ and $(0, 2, 2)$, and let Δ' denote the triangle with vertices $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$.

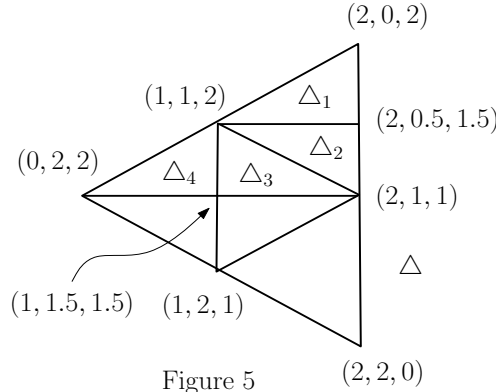


Figure 5

To estimate $\mu_1(C)$, based on Lemma 3.3, we assume that

$$\mathbf{x}_1 = (x_1, y_1, z_1) \in \text{int}(\Delta) \cap X$$

and therefore

$$x_1 + y_1 + z_1 = 4. \tag{4.3}$$

It can be shown that $C \cap (C + \mathbf{x}_1)$ is a centrally symmetric hexagon if $\mathbf{x}_1 \in \Delta^*$ and is a parallelogram if $\mathbf{x}_1 \in \text{int}(\Delta) \setminus \Delta^*$. By symmetry, we consider four cases $\mathbf{x}_1 \in \Delta_i$ by Lemmas 4.2-4.5, respectively.

Lemma 4.2. *If $\mathbf{x}_1 \in \Delta_1$, we have*

$$v(D_1(C, X)) \geq \frac{1}{6} \times \frac{9}{16}.$$

Proof. In this case $C + \mathbf{x}_1$ touches C at a parallelogram with vertices $(1, 0, 1)$, $(1, 2 - z_1, z_1 - 1)$, $(x_1 - 1, y_1, z_1 - 1)$ and $(x_1 - 1, 2 - x_1, 1)$. Then, by Lemma 2.3 and Lemma 2.4 it can be shown that $D_1(C, X)$ contains three tetrahedra T_1 , T_3 and T_5 and a prismoid P , where T_1 has vertices $(x_1 - 1, 1, 1)$, $(0, 1, 1)$, $(x_1 - 1, 2 - x_1, 1)$ and $(x_1 - 1, 1, 2 - x_1)$, T_3 has vertices $(1, 1, z_1 - 1)$, $(2 - z_1, 1, z_1 - 1)$, $(1, 2 - z_1, z_1 - 1)$ and $(1, 1, 0)$, T_5 has vertices $(1, 0, z_1 - 1)$, $(3 - z_1, 0, z_1 - 1)$, $(1, 2 - z_1, z_1 - 1)$ and $(1, 0, 1)$, and P has six vertices $(x_1 - 1, y_1, z_1 - 1)$, $(x_1 - 1, 1, z_1 - 1)$, $(x_1 - y_1, 1, z_1 - 1)$, $(x_1 - 1, 1, 1)$, $(x_1 - 1, 2 + y_1 - z_1, 1)$ and $(2 - 2y_1, 1, 1)$. In other words,

$$P = \{(x, y, z) : x \geq x_1 - 1, y \leq 1, z_1 - 1 \leq z \leq 1, (x - x_1) - (y - y_1) + (z - z_1) \leq -2\}.$$

Clearly, all T_1 , T_3 , T_5 and $T_1 \cap T_3$ are homothetic to the orthogonal tetrahedron T_0 which has vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ with ratios $x_1 - 1$, $z_1 - 1$, $2 - z_1$ and $1 - y_1$, respectively. The prismoid P

can be obtained by cutting off a tetrahedron T_7 with vertices $(x_1 - 1, 1, 1)$, $(2 - 2y_1, 1, 1)$, $(x_1 - 1, 2 + y_1 - z_1, 1)$ and $(x_1 - 1, 1, z_1 - y_1)$ from a tetrahedron T_6 with vertices $(x_1 - 1, 1, z_1 - 1)$, $(x_1 - y_1, 1, z_1 - 1)$, $(x_1 - 1, y_1, z_1 - 1)$ and $(x_1 - 1, 1, z_1 - y_1)$. Both T_6 and T_7 are homothetic to T_0 with ratios $1 - y_1$ and $z_1 - y_1 - 1$, respectively.

Since $(x_1, y_1, z_1) \in \Delta_1$, by a routine computation one can deduce

$$\begin{aligned} v(T_1 \cup T_3 \cup T_5 \cup P) &= \frac{1}{6} ((x_1 - 1)^3 + (z_1 - 1)^3 + (2 - z_1)^3 - (1 - y_1)^3 + (1 - y_1)^3 - (z_1 - y_1 - 1)^3) \\ &= \frac{1}{6} ((x_1 - 1)^3 + (z_1 - 1)^3 + (2 - z_1)^3 - (2z_1 + x_1 - 5)^3) \\ &\geq \frac{1}{6} \left(\left(\frac{1}{2} \right)^3 + \left(\frac{3}{4} \right)^3 + \left(\frac{1}{4} \right)^3 \right) \\ &= \frac{1}{6} \times \frac{9}{16}, \end{aligned}$$

where the equality holds if and only if $\mathbf{x}_1 = (\frac{3}{2}, \frac{3}{4}, \frac{7}{4})$. Thus, in this case we have

$$v(D_1(C, X)) \geq v(T_1 \cup T_3 \cup T_5 \cup P) \geq \frac{1}{6} \times \frac{9}{16}.$$

The lemma is proved. \square

Remark 4.2. It can be verified that $\mathbf{x}_1 = (\frac{3}{2}, \frac{3}{4}, \frac{7}{4}) \in \text{int}(H)$, where H was defined above Lemma 4.1. Therefore Lemma 4.2 can not be covered by Lemma 4.1.

Lemma 4.3. If $\mathbf{x}_1 \in \Delta_2$, we have

$$v(D_1(C, X)) \geq \frac{1}{6} \left(6 - 12\sqrt{\frac{6}{29}} \right).$$

Proof. In this case $C + \mathbf{x}_1$ touches C at the parallelogram with vertices $(1, 0, 1)$, $(1, 2 - z_1, z_1 - 1)$, $(x_1 - 1, y_1, z_1 - 1)$ and $(x_1 - 1, 2 - x_1, 1)$. Then, by Lemmas 2.3 and 2.4 it can be shown that $D_1(C, X)$ contains all T_1 , T_3 , T_5 and T_6 , where T_1 is the tetrahedron with vertices $(x_1 - 1, 1, 1)$, $(0, 1, 1)$, $(x_1 - 1, 2 - x_1, 1)$ and $(x_1 - 1, 1, 2 - x_1)$, T_3 is the tetrahedron with vertices $(1, 1, z_1 - 1)$, $(2 - z_1, 1, z_1 - 1)$, $(1, 2 - z_1, z_1 - 1)$ and $(1, 1, 0)$, T_5 is the tetrahedron with vertices $(1, 0, z_1 - 1)$, $(3 - z_1, 0, z_1 - 1)$, $(1, 2 - z_1, z_1 - 1)$ and $(1, 0, 1)$, and T_6 is the tetrahedron with vertices $(x_1 - 1, 1, z_1 - 1)$, $(x_1 - y_1, 1, z_1 - 1)$, $(x_1 - 1, y_1, z_1 - 1)$ and $(x_1 - 1, 1, z_1 - y_1)$.

Clearly, all T_1 , T_3 , T_5 , T_6 and $T_1 \cap T_3$ are homothetic to the orthogonal tetrahedron T_0 with ratios $x_1 - 1$, $z_1 - 1$, $2 - z_1$, $1 - y_1$ and $1 - y_1$, respectively. Thus, we have

$$\begin{aligned} v(T_1 \cup T_3 \cup T_5 \cup T_6) &= \frac{1}{6} ((x_1 - 1)^3 + (z_1 - 1)^3 - (1 - y_1)^3 + (2 - z_1)^3 + (1 - y_1)^3) \\ &= \frac{1}{6} ((x_1 - 1)^3 + (z_1 - 1)^3 + (2 - z_1)^3). \end{aligned} \quad (4.4)$$

Let U denote the rectangle with vertices $(1, 0, 0)$, $(1, 0, z_1 - 1)$, $(1, 1, z_1 - 1)$ and $(1, 1, 0)$ and define

$$G_1(C, X) = \left(\bigcup_{\mathbf{x} \in U} s(C, X, \mathbf{e}_1, \mathbf{x}) \right) \cap 2W. \quad (4.5)$$

We proceed to estimate the volume of $G_1(C, X)$. To this end, let S_0 to be the halfspace $\{(x, y, z) : x \geq 1\}$, let S_1 to be the halfspace $\{(x, y, z) : y \geq 0\}$, let S_2 to be the halfspace $\{(x, y, z) : z \geq 0\}$, let S_3 to be the halfspace $\{(x, y, z) : x \leq 2\}$, let S_4 to be the halfspace $\{(x, y, z) : y \leq 1\}$, let S_5 to be the halfspace $\{(x, y, z) : z \leq z_1 - 1\}$, let S_6 to be the halfspace $\{(x, y, z) : x + y - z \leq 2\}$, let S_7 to be the halfspace $\{(x, y, z) : (x - x_1) - (y - y_1) - (z - z_1) \leq 2\}$, and define

$$S = \bigcap_{i=0}^7 S_i.$$

Furthermore, we write $\mathbf{x}_t = (2, t, z_1 - 2)$ and $\mathbf{x}'_t = (2, t - 2, z_1 - 2)$ and define

$$P(t) = S \setminus \{(C + \mathbf{x}_t) \cup (C + \mathbf{x}'_t)\}.$$

For convenience, we write

$$F_5 = \{(x, y, z) : z = -1, |x| + |y| \leq 1\}.$$

If $\mathbf{x} \in X$ and $\text{int}(S) \cap (C + \mathbf{x}) \neq \emptyset$, by reducing the x -coordinate of \mathbf{x} until $C + \mathbf{x}$ touches C one can deduce $\mathbf{x}_0 \mathbf{x} \prec F_0$. In addition, if there are more than one $\mathbf{x} \in X$ satisfying $\text{int}(S) \cap (C + \mathbf{x}) \neq \emptyset$, they can not block each others in this reducing process. To see this, if $C + \mathbf{x}_2$ touches C at the interior of F_0 , and $C + \mathbf{x}_2$ blocks

$C + \mathbf{x}_3$ from touching C in \mathbf{e}_1 direction by $F_i + \mathbf{x}_2$, one can deduce contradiction by considering five cases $i = 0, 1, 2, 3$ and 4.

The $i = 0$ case is obvious. When $i = 1$ or 2, we have $\mathbf{x}_1\mathbf{x}_3 \prec F_5 + \mathbf{x}_1$, $z_3 \leq z_1 - 2$, $z_2 + 1 \leq z_3$, and therefore

$$z_2 \leq z_3 - 1 \leq z_1 - 3 \leq -1,$$

which contradicts the assumption that $\text{int}(S) \cap (C + \mathbf{x}_2) \neq \emptyset$. When $i = 3$, we get

$$\begin{cases} |y_2| + |z_2| \leq 2 \\ (x_3 - 2) - (y_3 - y_2) - (z_3 - z_2) = 4 \end{cases}$$

and therefore

$$x_3 - y_3 - z_3 = 6 - (y_2 + z_2) \geq 4,$$

which contradicts the assumption that $\text{int}(S) \cap (C + \mathbf{x}_3) \neq \emptyset$. When $i = 4$, we get

$$\begin{cases} |y_2| + |z_2| \leq 2 \\ (x_3 - 2) + (y_3 - y_2) - (z_3 - z_2) = 4 \end{cases}$$

and therefore

$$x_3 + y_3 - z_3 = 6 + y_2 - z_2 \geq 4,$$

which contradicts the assumption that $\text{int}(S) \cap (C + \mathbf{x}_3) \neq \emptyset$.

Then, by increasing the z -coordinate of \mathbf{x} until $C + \mathbf{x}$ touches $C + \mathbf{x}_1$, it can be shown that

$$v(G_1(C, X)) \geq \min_t v(P(t)). \quad (4.6)$$

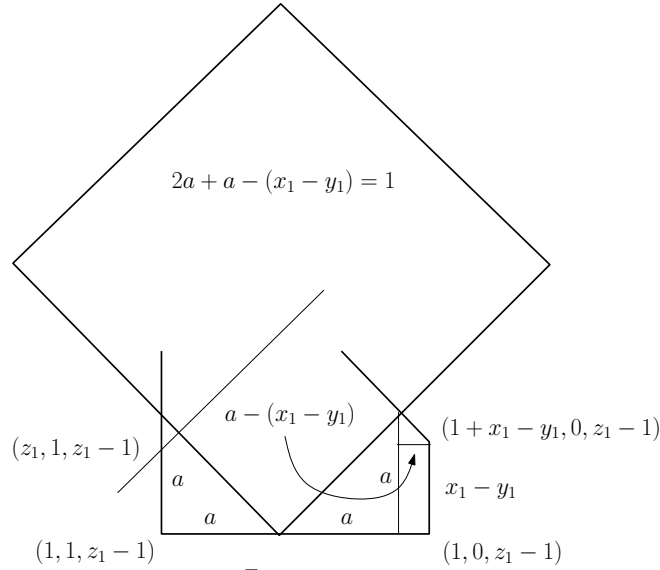


Figure 6

It can be shown that the three hyperplanes $y = 0$, $z = z_1 - 1$ and $(x - x_1) - (y - y_1) - (z - z_1) = 2$ intersect at $(1 + x_1 - y_1, 0, z_1 - 1)$, and the three hyperplanes $y = 1$, $z = z_1 - 1$ and $x + y - z = 2$ meet at $(z_1, 1, z_1 - 1)$. Based on Figure 6, which illustrates a possible $C + \mathbf{x}$ touching $C + \mathbf{x}_1$ in the half plane $\{(x, y, z) : z = z_1 - 1, x \geq 1\}$, by studying the derivative of $v(P(t))$ in terms of parts of surface areas of $P(t)$ it can be shown that

$$v(P(t)) \geq \frac{1}{6} \begin{cases} 2 \left(\frac{1}{2}\right)^3, & \text{if } x_1 - y_1 \geq \frac{1}{2}, z_1 - 1 \geq \frac{1}{2}; \\ 2(z_1 - 1)^3, & \text{if } x_1 - y_1 \geq \frac{1}{2}, z_1 - 1 \leq \frac{1}{2}; \\ 2 \left(\frac{1+x_1-y_1}{3}\right)^3, & \text{if } x_1 - y_1 \leq \frac{1}{2}, z_1 - 1 \geq \frac{1}{3}(1+x_1-y_1); \\ 2(z_1 - 1)^3, & \text{if } x_1 - y_1 \leq \frac{1}{2}, z_1 - 1 \leq \frac{1}{3}(1+x_1-y_1). \end{cases}$$

Thus, we get

$$\min_t v(P(t)) \geq \frac{1}{6} \min \left\{ 2 \left(\frac{1+x_1-y_1}{3}\right)^3, 2(z_1 - 1)^3, 2 \left(\frac{1}{2}\right)^3 \right\}. \quad (4.7)$$

By routine computations it can be shown that, when $\mathbf{x}_1 \in \Delta_2$,

$$(x_1 - 1)^3 + (z_1 - 1)^3 + (2 - z_1)^3 + 2 \left(\frac{1+x_1-y_1}{3}\right)^3 \geq 6 - 12\sqrt{\frac{6}{29}}, \quad (4.8)$$

$$(x_1 - 1)^3 + (z_1 - 1)^3 + (2 - z_1)^3 + 2(z_1 - 1)^3 \geq \frac{6}{5 + 2\sqrt{6}} \geq 6 - 12\sqrt{\frac{6}{29}} \quad (4.9)$$

and

$$(x_1 - 1)^3 + (z_1 - 1)^3 + (2 - z_1)^3 + 2\left(\frac{1}{2}\right)^3 \geq \frac{11 + 2\sqrt{2}}{12 + 8\sqrt{2}} \geq 6 - 12\sqrt{\frac{6}{29}}. \quad (4.10)$$

Thus, in this case it can be deduced from (4.4), (4.6)-(4.10) that

$$\begin{aligned} v(D_1(C, X)) &\geq v(T_1 \cup T_3 \cup T_5 \cup T_6) + v(G_1(C, X)) \\ &\geq v(T_1 \cup T_3 \cup T_5 \cup T_6) + \min_t \{v(P(t))\} \\ &= \frac{1}{6} \left((x_1 - 1)^3 + (z_1 - 1)^3 + (2 - z_1)^3 + \min \left\{ 2 \left(\frac{1 + x_1 - y_1}{3} \right)^3, 2(z_1 - 1)^3, 2 \left(\frac{1}{2} \right)^3 \right\} \right) \\ &\geq \frac{1}{6} \left(6 - 12\sqrt{\frac{6}{29}} \right). \end{aligned}$$

Lemma 4.3 is proved. \square

Remark 4.3. We notice that

$$6 - 12\sqrt{\frac{6}{29}} \approx 0.541694086 \dots$$

We define

$$P_1 = \{(x, y, z) : 0 \leq x < 2, 0 < y < 1, 0 < z < 1, x - y + z < 2, x - y - z < 1, x + y - z < 2\},$$

$$P_2 = \{(x, y, z) : 1 \leq x < 3, -1 < y < 2, -1 < z < 2, x - y + z < 4, x - y - z < 3, x + y - z < 4\}$$

and

$$D'_1(C, X) = D_1(C, X) \cap P_1.$$

It follows by Lemma 3.3 that X has a point $\mathbf{x}_1 = (x_1, y_1, z_1)$ which belongs to $\text{int}(\Delta)$. For convenience, we write $\mathbf{x}_0 = \mathbf{o}$ and define

$$X' = (X \cap P_2) \cup \{\mathbf{x}_0, \mathbf{x}_1\}. \quad (4.11)$$

It can be verified that

$$D'_1(C, X') = D'_1(C, X).$$

By the minimal assumptions on $v(D_1(C, X))$ we can get

$$\left((C + \mathbf{x}_1) \cup C \right) \cap \left(\bigcup_{\mathbf{x} \in X' \setminus \{\mathbf{x}_0, \mathbf{x}_1\}} (C + \mathbf{x}) \right) \neq \emptyset. \quad (4.12)$$

For convenience, we write

$$J = \left((C + \mathbf{x}_1) \cup C \right) \cap \left(\bigcup_{\mathbf{x} \in X' \setminus \{\mathbf{x}_0, \mathbf{x}_1\}} (C + \mathbf{x}) \right).$$

Let F_i be the facets defined just above (3.9). In particular, F_0 is the square facet $\{(x, y, z) : x = 1, |y| + |z| \leq 1\}$ of C , and F_3 is the triangular facet $\{(x, y, z) : \max\{|x|, |y|, |z|\} \leq 1, x - y - z = 2\}$. It follows by (4.11) and (4.12) that

$$J \cap (\text{int}(F_0) \cup \{\text{int}(F_3) + \mathbf{x}_1\} \cup \{F_0 + \mathbf{x}_1\}) \neq \emptyset. \quad (4.13)$$

Lemma 4.4. If $\mathbf{x}_1 \in \Delta_4 \cap H$, then we have

$$v(D_1(C, X)) \geq \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right).$$

Proof. In this case $C + \mathbf{x}_1$ touches C at a parallelogram with vertices $(0, 1, 1)$, $(2 - z_1, 1, z_1 - 1)$, $(x_1, y_1 - 1, z_1 - 1)$ and $(x_1 - z_1 - 2, y_1 - 1, 1)$. Then, by routine arguments based on Lemmas 2.3 and 2.4 it can be shown that $D'_1(C, X')$ contains both T_2 and T_3 , where T_2 is the tetrahedron with vertices $(1, y_1 - 1, 1)$, $(2 - y_1, y_1 - 1, 1)$, $(1, 0, 1)$ and $(1, y_1 - 1, 2 - y_1)$, and T_3 is the tetrahedron with vertices $(1, 1, z_1 - 1)$, $(2 - z_1, 1, z_1 - 1)$, $(1, 2 - z_1, z_1 - 1)$ and $(1, 1, 0)$. Clearly all T_2 , T_3 and $T_2 \cap T_3$ are homothetic to the orthogonal tetrahedron T_0 with ratios $z_1 - 1$, $y_1 - 1$ and $1 - x_1$, respectively. Thus we have

$$v(T_2 \cup T_3) = \frac{1}{6} ((z_1 - 1)^3 + (y_1 - 1)^3 - (1 - x_1)^3) \geq \frac{1}{6} \times \frac{1}{4}. \quad (4.14)$$

To estimate $v(D'_1(C, X'))$, based on (4.13) we consider three cases.

Case 1. $J \cap (\text{int}(F_0) \cup (\text{int}(F_3) + \mathbf{x}_1)) = \emptyset$. By the assumption on X' we have

$$x \geq x_1 + 2$$

for all $\mathbf{x} = (x, y, z) \in X' \setminus \{\mathbf{x}_0, \mathbf{x}_1\}$. Let T^* denote the tetrahedron with vertices $(1, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(1 + x_1, \frac{1}{2}, \frac{1}{2})$. Clearly we have

$$T^* \subset D'_1(C, X')$$

and

$$v(T^*) = \frac{1}{6} x_1.$$

Thus, applying (4.14) and (4.3), we get

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup T^*) \\ &= \frac{1}{6} ((z_1 - 1)^3 + (y_1 - 1)^3 - (1 - x_1)^3 + x_1) \\ &\geq \frac{1}{6} \left(2 \left(1 - \frac{x_1}{2} \right)^3 - (1 - x_1)^3 + x_1 \right) \\ &\geq \frac{1}{6} \left(2 \left(1 - \frac{\alpha}{2} \right)^3 - (1 - \alpha)^3 + \alpha \right) \\ &> \frac{1}{6} \times 1. \end{aligned} \tag{4.15}$$

Case 2. $\text{int}(F_3 + \mathbf{x}_1) \cap J \neq \emptyset$. Assume that $C + \mathbf{x}_2$, where $\mathbf{x}_2 = (x_2, y_2, z_2) \in X'$, touches $C + \mathbf{x}_1$ at some interior points of $F_3 + \mathbf{x}_1$. Clearly we have

$$y_2 \geq y_1 - 2 \quad \text{and} \quad z_2 \geq z_1 - 2 \geq y_1 - 2.$$

Then we consider two subcases.

Subcase 2.1. $y_2 \geq z_1 - 2$. Let us define T_8 to be the tetrahedron with vertices $(1, 1, z_1 - 1)$, $(2 + x_1 - y_1, 1, z_1 - 1)$, $(1, y_1 - x_1, z_1 - 1)$ and $(1, 1, 2 - 2x_1)$, and define T_9 to be the tetrahedron with vertices $(1, y_1 - 1, 1)$, $(2 + x_1 - z_1, y_1 - 1, 1)$, $(1, 2 - 2x_1, 1)$ and $(1, y_1 - 1, z_1 - x_1)$.

If there is a point $\mathbf{x} = (x, y, z)$ satisfying both

$$\text{int}(T_8) \cap (C + \mathbf{x}) \neq \emptyset$$

and

$$(\text{int}(C) + \mathbf{x}) \cap (C + \mathbf{x}_i) = \emptyset, \quad i = 0, 1, 2,$$

by Lemma 2.3 and Lemma 2.4 it can be verified that neither $C + \mathbf{x}_2$ nor $C + \mathbf{x}_1$ can block $C + \mathbf{x}$ from moving in $-\mathbf{e}_1$ direction. Thus, one can assume that $\mathbf{x} = (2, y, z)$ and therefore $C + \mathbf{x}$ touches C at some interior points of F_0 . Then the point $(1, y_2 + 1, z_1 - 2)$, which is one unit below $(1, y_2 + 1, z_1 - 1)$ in z direction, and the origin \mathbf{o} are on the same side of the hyperplane $\{(x, y, z) : x + y - z = 2\}$. Therefore we get

$$y_2 < 2 + (z_1 - 2) - 1 - 1 = z_1 - 2,$$

which contradicts the assumption on y_2 . Thus we have

$$\text{int}(T_i) \cap (C + X') = \emptyset, \quad i = 8, 9.$$

Clearly, both T_8 and T_9 are homothetic to T_0 , with ratios $1 + x_1 - y_1$ and $1 + x_1 - z_1$ respectively. Moreover, they are disjoint. Since both $1 + x_1 - y_1$ and $1 + x_1 - z_1$ are not larger than x_1 , by Lemma 2.3 it follows that

$$T_8 \cup T_9 \subset D'_1(C, X').$$

Then, applying (4.14) and the assumption on \mathbf{x}_1 we get

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup T_8 \cup T_9) \\ &= \frac{1}{6} ((z_1 - 1)^3 + (y_1 - 1)^3 - (1 - x_1)^3 + (1 + x_1 - y_1)^3 + (1 + x_1 - z_1)^3) \\ &\geq \frac{1}{6} \left(2 \left(1 - \frac{x_1}{2} \right)^3 - (1 - x_1)^3 + 2 \left(\frac{3x_1}{2} - 1 \right)^3 \right) \\ &\geq \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right), \end{aligned} \tag{4.16}$$

where all the equalities hold if and only if $\mathbf{x}_1 = \left(\frac{2}{3} + \frac{2}{3}\sqrt{\frac{1}{10}}, \frac{5}{3} - \frac{1}{3}\sqrt{\frac{1}{10}}, \frac{5}{3} - \frac{1}{3}\sqrt{\frac{1}{10}}\right)$ and $\mathbf{x}_2 = \left(2, \frac{1}{3} - \frac{2}{3}\sqrt{\frac{1}{10}}, \frac{1}{3} - \frac{2}{3}\sqrt{\frac{1}{10}}\right)$.

Subcase 2.2. $y_2 \leq z_1 - 2$. Then the assumption $\text{int}(F_3 + \mathbf{x}_1) \cap J \neq \emptyset$ implies

$$y_1 - 2 \leq y_2 \leq z_1 - 2. \quad (4.17)$$

Let T'_0 to be the orthogonal tetrahedron with vertices $(1, y_2 + 1, z_1 - 1)$, $(2, y_2 + 1, z_1 - 1)$, $(1, y_2 + 2, z_1 - 1)$ and $(1, y_2 + 1, z_1 - 2)$, let S_2, S_8, S_9, S_{10} and S_{11} denote the halfspaces $\{(x, y, z) : z \geq 0\}$, $\{(x, y, z) : x \leq x_1 + 1\}$, $\{(x, y, z) : z \geq z_2 - 1\}$, $\{(x, y, z) : x - y - z \geq 1\}$ and $\{(x, y, z) : x - y + z \geq 2\}$, respectively, and define

$$P_3 = P_1 \cap T'_0 \cap S_8 \cap S_9.$$

It can be verified by Lemma 2.3 and Lemma 2.4 that

$$P_3 \subseteq D'_1(C, X').$$

By $x_2 - y_2 + z_2 \leq 4$, $(x_2 - x_1) - (y_2 - y_1) - (z_2 - z_1) = 4$ and $x_1 + y_1 + z_1 = 4$ one can deduce $z_2 \leq 2 - x_1$. Then by routine computations based on (4.17) it can be shown that

$$\begin{aligned} v(T'_0 \setminus \{S_2 \cap S_9\}) &\leq \max \left\{ \frac{1}{6}(2 - z_1)^3, \frac{1}{6}(1 - z_1 + z_2)^3 \right\} \leq \frac{1}{6}(y_1 - 1)^3, \\ v(T'_0 \cap S_8 \cap S_{10}) &\leq \frac{1}{2} \left(\frac{1 - z_1 - y_2}{\sqrt{2}} \right)^2 x_1 \leq \frac{1}{6} \times \frac{3}{2}(2z_1 - 3)^2 x_1, \\ v(T'_0 \cap S_{11}) &\leq \frac{1}{6} \times 2 \left(\frac{z_1 - y_2 - 2}{2} \right)^3 \leq \frac{1}{6} \times \frac{1}{4}(z_1 - y_1)^3, \end{aligned}$$

and therefore

$$v(P_3) \geq \frac{1}{6} \left(1 - (1 - x_1)^3 - (1 + y_2)^3 - (y_1 - 1)^3 - \frac{3}{2}(2z_1 - 3)^2 x_1 - \frac{1}{4}(z_1 - y_1)^3 \right).$$

Thus, by (4.14), (4.17) and the assumptions on \mathbf{x}_1 and \mathbf{x}_2 we obtain

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_3) \\ &\geq \frac{1}{6} \left(1 + (z_1 - 1)^3 - 2(1 - x_1)^3 - (y_2 + 1)^3 - \frac{3}{2}(2z_1 - 3)^2 x_1 - \frac{1}{4}(z_1 - y_1)^3 \right) \\ &\geq \frac{1}{6} \left(1 - 2(1 - x_1)^3 - \frac{3}{2}(2z_1 - 3)^2 x_1 - \frac{1}{4}(2\beta - 3)^3 \right) \\ &\geq \frac{1}{6} \left(1 - \frac{3}{2}(2\beta - 3)^2 - \frac{1}{4}(2\beta - 3)^3 \right) \\ &> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9}\sqrt{\frac{1}{10}} \right). \end{aligned} \quad (4.18)$$

As a conclusion of (4.16) and (4.18), in this case we have

$$v(D'_1(C, X')) \geq \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9}\sqrt{\frac{1}{10}} \right).$$

Case 3. $\text{int}(F_0) \cap J \neq \emptyset$. Assume that $C + \mathbf{x}_2$, where $\mathbf{x}_2 = (2, y_2, z_2) \in X'$, touches C at some interior points of F_0 . We divide the region $\{(y_2, z_2) : -1 \leq y_2 \leq 2, -1 \leq z_2 \leq 2\}$ into nine parts as illustrated in Figure 7 and consider the following subcases.

Subcase 3.1. $y_2 \geq z_1 - 2$ and $z_2 \geq z_1 - 2$. Then, the two translates $C + \mathbf{x}_1$ and $C + \mathbf{x}_2$ are separated by the hyperplane $(x - x_1) - (y - y_1) - (z - z_1) = 2$. Let T_8 and T_9 be the tetrahedra defined in Subcase 2.1, we also have

$$T_8 \cup T_9 \subset D'_1(C, X').$$

If, on the contrary, there is a translate $C + \mathbf{x}$ satisfying both

$$(C + \mathbf{x}) \cap \text{int}(T_8) \neq \emptyset$$

and

$$(C + \mathbf{x}) \cap (\text{int}(C) + \mathbf{x}_i) = \emptyset, \quad i = 0, 1, 2,$$

by reducing the x -coordinate of \mathbf{x} one can assume that $C + \mathbf{x}$ touches $(C + \mathbf{x}_0) \cup (C + \mathbf{x}_1) \cup (C + \mathbf{x}_2)$ at its boundary. By considering cases with respect to the facet of the touching points, one can reach contradictions

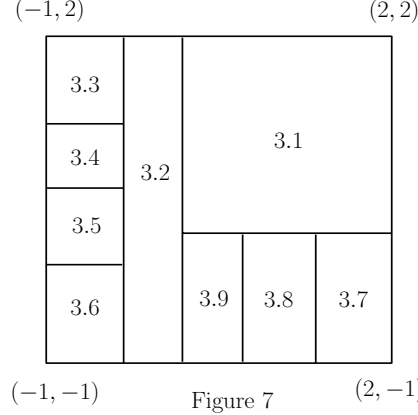


Figure 7

one by one. For example, if $C + \mathbf{x}$ touches $C + \mathbf{x}_2$ at some interior point of $F_1 + \mathbf{x}_2$, $x < x_1 + 2$, and $\mathbf{x}_1 \mathbf{x} \prec F_5 + \mathbf{x}_1$, we get

$$\begin{cases} (x-2) + (y-y_2) + (z-z_2) = 4, \\ x < x_1 + 2, \\ z \leq z_1 - 2, \end{cases}$$

and therefore

$$z_2 = x + y + z - y_2 - 6 < (x_1 + z_1) + (y - y_2) - 6 < -1,$$

which contradicts the assumption that $\mathbf{x}_2 \in P_2$. In fact, this example is the only nontrivial case.

Similar to Subcase 2.1 we obtain

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup T_8 \cup T_9) \\ &= \frac{1}{6} ((y_1 - 1)^3 + (z_1 - 1)^3 - (1 - x_1)^3 + (1 + x_1 - y_1)^3 + (1 + x_1 - z_1)^3) \\ &\geq \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right), \end{aligned} \tag{4.19}$$

where all the equalities hold if and only if $\mathbf{x}_1 = \left(\frac{2}{3} + \frac{2}{3} \sqrt{\frac{1}{10}}, \frac{5}{3} - \frac{1}{3} \sqrt{\frac{1}{10}}, \frac{5}{3} - \frac{1}{3} \sqrt{\frac{1}{10}} \right)$ and $\mathbf{x}_2 = \left(2, \frac{1}{3} - \frac{2}{3} \sqrt{\frac{1}{10}}, \frac{1}{3} - \frac{2}{3} \sqrt{\frac{1}{10}} \right)$.

Subcase 3.2. $y_1 - 2 \leq y_2 \leq z_1 - 2$. Let $P_3, T'_0, S_2, S_8, S_9, S_{10}$ and S_{11} be the polytopes defined in Subcase 2.2, here we also have

$$P_3 \subseteq D'_1(C, X').$$

If, on the contrary, there is a translate $C + \mathbf{x}$ satisfying both

$$(C + \mathbf{x}) \cap \text{int}(P_3) \neq \emptyset$$

and

$$(C + \mathbf{x}) \cap (\text{int}(C) + \mathbf{x}_i) = \emptyset, \quad i = 0, 1, 2,$$

by reducing the x -coordinate of \mathbf{x} one can assume that $C + \mathbf{x}$ touches $(C + \mathbf{x}_0) \cup (C + \mathbf{x}_1) \cup (C + \mathbf{x}_2)$ at its boundary. By considering cases with respect to the facet of the touching points, one can reach contradictions one by one. For example, if $C + \mathbf{x}$ touches $C + \mathbf{x}_2$ at some interior point of $F_1 + \mathbf{x}_2$, $x < x_1 + 2$, and $\mathbf{x}_1 \mathbf{x} \prec F_3 + \mathbf{x}_1$, we have

$$\begin{cases} (x-2) + (y-y_2) + (z-z_2) = 4, \\ x < x_1 + 2, \\ (x-x_1) - (y-y_1) - (z-z_1) \geq 4, \end{cases}$$

and therefore

$$z_2 = x + (y + z) - y_2 - 6 \leq 2x - x_1 + y_1 + z_1 - y_2 - 10 < 4 + x_1 + y_1 + z_1 - y_1 + 2 - 10 < -y_1 \leq -1,$$

which contradicts the assumption that $\mathbf{x}_2 \in P_2$.

If $z_1 - z_2 > 2$ and $2 - x_1 - (y_2 + 1 - y_1) - (z_2 + 1 - z_1) < 2$, it can be deduced that $y_1 - x_1 < y_2 \leq z_1 - 2$ and $y_1 < 1$, which contradicts the assumption $\mathbf{x}_1 \in \Delta_4 \cap H$. Thus we have

$$2 - x_1 - (y_2 + 1 - y_1) - (z_2 + 1 - z_1) \geq 2,$$

$$1 - z_1 + z_2 \leq y_1 - x_1 - y_2 - 1 \leq 1 - x_1$$

and therefore

$$v(T'_0 \setminus \{S_2 \cap S_9\}) \leq \max \left\{ \frac{1}{6}(2 - z_1)^3, \frac{1}{6}(1 - z_1 + z_2)^3 \right\} \leq \frac{1}{6}((2 - z_1)^3 + (1 - x_1)^3).$$

In addition, similar to Subcase 2.2, we have

$$v(T'_0 \cap S_8 \cap S_{10}) \leq \frac{1}{2} \left(\frac{1 - z_1 - y_2}{\sqrt{2}} \right)^2 x_1 \leq \frac{1}{6} \times \frac{3}{2} (2z_1 - 3)^2 x_1,$$

$$v(T'_0 \cap S_{11}) \leq \frac{1}{6} \times 2 \left(\frac{z_1 - y_2 - 2}{2} \right)^3 \leq \frac{1}{6} \times \frac{1}{4} (z_1 - y_1)^3$$

and therefore

$$\begin{aligned} v(P_3) &\geq \frac{1}{6} \left(1 - 2(1 - x_1)^3 - (1 + y_2)^3 - (2 - z_1)^3 - \frac{3}{2}(2z_1 - 3)^2 x_1 - \frac{1}{4}(z_1 - y_1)^3 \right) \\ &\geq \frac{1}{6} \left(1 - 2(1 - x_1)^3 - (z_1 - 1)^3 - (2 - z_1)^3 - \frac{3}{2}(2z_1 - 3)^2 x_1 - \frac{1}{4}(z_1 - y_1)^3 \right). \end{aligned}$$

Thus, by (4.14) and the assumptions on \mathbf{x}_1 we obtain

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_3) \\ &\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 - 3(1 - x_1)^3 - (2 - z_1)^3 - \frac{3}{2}(2z_1 - 3)^2 x_1 - \frac{1}{4}(z_1 - y_1)^3 \right) \\ &\geq \frac{1}{6} \left(1 - 3(1 - x_1)^3 - \frac{3}{2}(2z_1 - 3)^2 x_1 - \frac{1}{4}(z_1 - y_1)^3 \right) \\ &\geq \frac{1}{6} \left(1 - \frac{3}{2}(2\beta - 3)^2 - \frac{1}{4}(2\beta - 3)^3 \right) \\ &> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \end{aligned} \tag{4.20}$$

Subcase 3.3. $-1 \leq y_2 \leq y_1 - 2$ and $z_2 \geq 1.25$. By the assumption $\mathbf{x}_2 = (2, y_2, z_2) \in P_2$ we get

$$2 - y_2 + z_2 < 4.$$

Thus one can deduce

$$1.25 \leq z_2 < y_2 + 2 \leq y_1,$$

which implies

$$-0.75 \leq y_2$$

and

$$-0.5 \leq y_2 + z_2 - 1 \leq 2y_1 - 3 \leq 1 - \alpha.$$

Let T_0^* denote the orthogonal tetrahedron with vertices $(1, y_2 + 1, z_2 - 1)$, $(2, y_2 + 1, z_2 - 1)$, $(1, y_2, z_2 - 1)$ and $(1, y_2 + 1, z_2)$, let P_1 be the polytope defined above (4.11), let S_8 be the halfspace defined in Subcase 2.2, and define

$$P_4 = P_1 \cap T_0^* \cap S_8.$$

By Lemma 2.3 and Lemma 2.4 it can be verified that

$$P_4 \subset D'_1(C, X').$$

We recall $S_8 = \{(x, y, z) : x \leq x_1 + 1\}$ and $S_{10} = \{(x, y, z) : x - y - z \geq 1\}$, and define $S'_6 = \{(x, y, z) : x + y - z \geq 2\}$. By routine computations we have

$$v(T_0^* \cap S'_6) = \frac{1}{6} \times \frac{1}{4} (2 + y_2 - z_2)^3,$$

$$v(T_0^* \cap S_8 \cap S_{10}) \leq \frac{1}{6} \times \frac{3}{2} (y_2 + z_2 - 1)^2 x_1,$$

and

$$v(P_4) \geq \frac{1}{6} \left(1 - (z_2 - 1)^3 - \frac{1}{4}(2 + y_2 - z_2)^3 + y_2^3 - \frac{3}{2}(y_2 + z_2 - 1)^2 x_1 - (1 - x_1)^3 \right).$$

Then, by the assumptions on \mathbf{x}_1 and \mathbf{x}_2 and by (4.14), we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_4) \\
&\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 + (z_1 - 1)^3 - 2(1 - x_1)^3 - (z_2 - 1)^3 - \frac{1}{4}(2 + y_2 - z_2)^3 + y_2^3 \right. \\
&\quad \left. - \frac{3}{2}(y_2 + z_2 - 1)^2 x_1 \right) \\
&\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 + (z_1 - 1)^3 - 2(1 - x_1)^3 - (y_2 + 1)^3 + y_2^3 - \frac{3}{2}(y_2 + z_2 - 1)^2 \right) \\
&\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 + (z_1 - 1)^3 - 2(1 - x_1)^3 - \left(\frac{1}{4}\right)^3 - \left(\frac{3}{4}\right)^3 - \frac{3}{2}(y_2 + z_2 - 1)^2 \right) \\
&\geq \frac{1}{6} \left(1 + \frac{1}{4} - (1 - \alpha)^3 - \left(\frac{1}{4}\right)^3 - \left(\frac{3}{4}\right)^3 - \frac{3}{2} \left(\frac{1}{2}\right)^2 \right) \\
&> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \tag{4.21}
\end{aligned}$$

Subcase 3.4. $-1 \leq y_2 \leq y_1 - 2$ and $1 \leq z_2 \leq 1.25$. Then by $(2, y_2, z_2) \in P_2$ one can deduce

$$1 \leq z_2 \leq \min\{1.25, y_2 + 2\}.$$

Let P_3 be the polytope defined in Subcase 2.2 and let P_4 be the polytope defined in Subcase 3.3. It can be shown that

$$\text{int}(P_3) \cap \text{int}(P_4) = \emptyset$$

and, by Lemma 2.3 and Lemma 2.4,

$$P_i \subset D'_1(C, X'), \quad i = 3, 4.$$

By routine computations it can be shown that

$$v(P_3) \geq \frac{1}{6} \left(1 - (y_2 + 1)^3 - \frac{1}{4}(z_1 - y_2 - 2)^3 - (1 - z_1 + z_2)^3 - \frac{3}{2}(1 - y_2 - z_1)^2(z_1 - z_2) - (1 - x_1)^3 \right)$$

and

$$v(P_4) \geq \frac{1}{6} \left(1 - (z_2 - 1)^3 - \frac{1}{4}(2 + y_2 - z_2)^3 + y_2^3 - \frac{3}{2}(y_2 + z_2 - 1)^2(y_2 + 1) - (1 - x_1)^3 \right).$$

If $y_2 \geq -0.8$, we have

$$\begin{aligned}
-0.8 &\leq y_2 \leq y_1 - 2 \leq -\frac{\alpha}{2}, \\
2 + y_2 - z_2 &\leq 1 + y_2 \leq 1 - \frac{\alpha}{2}
\end{aligned}$$

and

$$-0.8 \leq y_2 + z_2 - 1 \leq \frac{1}{4} - \frac{\alpha}{2}.$$

Then we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_4) \\
&\geq \frac{1}{6} \left(\frac{5}{4} - \left(\frac{1}{4}\right)^3 - \frac{1}{4} \left(1 - \frac{\alpha}{2}\right)^3 + y_2^3 - \frac{3}{2} y_2^2 (y_2 + 1) - (1 - \alpha)^3 \right) \\
&\geq \frac{1}{6} \left(\frac{5}{4} - \left(\frac{1}{4}\right)^3 - \frac{1}{4} \left(1 - \frac{\alpha}{2}\right)^3 - \left(\frac{4}{5}\right)^3 - \frac{3}{2} \left(\frac{4}{5}\right)^2 \frac{1}{5} - (1 - \alpha)^3 \right) \\
&> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \tag{4.22}
\end{aligned}$$

When $-1 \leq y_2 \leq -0.8$ and $1 \leq z_2 \leq 1.25$, we have

$$z_1 - y_2 - 2 \leq \beta - 1$$

and

$$0 \leq 1 - y_2 - z_1 \leq 2 - z_1 \leq \frac{1}{2}.$$

Then we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_3) \\
&\geq \frac{1}{6} \left(\frac{5}{4} - \left(\frac{1}{5} \right)^3 - \frac{1}{4} (\beta - 1)^3 - (1 - (z_1 - z_2))^3 - \frac{3}{2} \left(\frac{1}{2} \right)^2 (z_1 - z_2) - (1 - \alpha)^3 \right) \\
&\geq \frac{1}{6} \left(\frac{5}{4} - \left(\frac{1}{5} \right)^3 - \frac{1}{4} (\beta - 1)^3 - \left(\frac{3}{4} \right)^3 - \frac{3}{2} \left(\frac{1}{2} \right)^2 \frac{1}{4} - (1 - \alpha)^3 \right) \\
&> \frac{1}{6} \times \frac{1}{2}.
\end{aligned} \tag{4.23}$$

As a conclusion of (4.22) and (4.23), in this subcase we have

$$v(D'_1(C, X')) > \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right).$$

Subcase 3.5. $-1 \leq y_2 \leq y_1 - 2$ and $y_1 - 2 \leq z_2 \leq 1$. Then we have

$$P_3 \subset D'_1(C, X'),$$

where P_3 was defined in Subcase 2.2. However, this time we have

$$z_1 - z_2 \geq z_1 - 1$$

and

$$v(P_3) \geq \frac{1}{6} \left(1 - (1 + y_2)^3 - \frac{1}{4} (z_1 - y_2 - 2)^3 - (2 - z_1)^3 - \frac{3}{4} (1 - y_2 - z_1)^2 (4 + y_2 - z_1) - (1 - x_1)^3 \right).$$

By routine computations one can deduce

$$(1 + y_2)^3 + \frac{1}{4} (z_1 - y_2 - 2)^3 \leq \max \left\{ \frac{1}{4} (z_1 - 1)^3, (z_1 - 1)^3 \right\} \leq (z_1 - 1)^3$$

and

$$(1 - y_2 - z_1)^2 (4 + y_2 - z_1) \leq \max \{ (2 - z_1)^2 (3 - z_1), (1 - x_1)^2 (2 + y_1 - z_1) \} \leq \left(\frac{1}{2} \right)^2 \frac{3}{2}.$$

Thus, we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_3) \\
&\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 + (z_1 - 1)^3 - 2(1 - x_1)^3 - (1 + y_2)^3 - \frac{1}{4} (z_1 - y_2 - 2)^3 \right. \\
&\quad \left. - (2 - z_1)^3 - \frac{3}{4} (1 - y_2 - z_1)^2 (4 + y_2 - z_1) \right) \\
&\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 - 2(1 - x_1)^3 - (2 - z_1)^3 - \frac{3}{4} \left(\frac{1}{2} \right)^2 \frac{3}{2} \right) \\
&\geq \frac{1}{6} \left(1 + (2 - \beta)^3 - 2(1 - \alpha)^3 - \left(\frac{1}{2} \right)^3 - \frac{3}{4} \left(\frac{1}{2} \right)^2 \frac{3}{2} \right) \\
&> \frac{1}{6} \times \frac{1}{2}.
\end{aligned} \tag{4.24}$$

Subcase 3.6. $-1 \leq y_2 \leq y_1 - 2$ and $z_2 \leq y_1 - 2$. First, by $\mathbf{x}_2 \in P_2$ and $\mathbf{x}_1 \in \Delta_4 \cap H$ one can deduce

$$y_2 + z_2 \geq -1, \tag{4.25}$$

$$\frac{\alpha}{2} - 1 \leq 1 - y_1 \leq -1 - z_2 \leq y_2 \leq y_1 - 2 \leq -\frac{\alpha}{2} \tag{4.26}$$

and

$$\frac{\alpha}{2} - 1 \leq 1 - y_1 \leq -1 - y_2 \leq z_2 \leq y_1 - 2 \leq -\frac{\alpha}{2}. \tag{4.27}$$

Let T_0° denote the tetrahedron with vertices $(1, y_2 + 1, z_2 + 1)$, $(2, y_2 + 1, z_2 + 1)$, $(1, y_2, z_2 + 1)$ and $(1, y_2 + 1, z_2)$, let S_8 and S_{11} be the halfspaces defined in Subcase 2.2, and define

$$P_5 = P_1 \cap T_0^\circ \cap S_8.$$

It can be verified by Lemma 2.3 and Lemma 2.4 that

$$P_5 \subset D'_1(C, X').$$

By routine computations it can be deduced that

$$v(T_0^\circ \cap S_8 \cap S_{11}) \leq \frac{1}{2} \left(\frac{y_2 - z_2}{\sqrt{2}} \right)^2 x_1 = \frac{1}{4} (y_2 - z_2)^2 x_1$$

and therefore

$$v(P_5) \geq \frac{1}{6} \left(1 + y_2^3 + z_2^3 - \frac{3}{2} (y_2 - z_2)^2 x_1 - (1 - x_1)^3 \right).$$

Thus, by (4.26), (4.27) and (4.25) we get

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_5) \\ &\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 + (z_1 - 1)^3 - 2(1 - x_1)^3 + y_2^3 + z_2^3 - \frac{3}{2} (y_2 - z_2)^2 x_1 \right) \\ &\geq \frac{1}{6} \left(\frac{5}{4} - (1 - \alpha)^3 + y_2^3 + z_2^3 - \frac{3}{2} (1 - \alpha)^2 \right) \\ &\geq \frac{1}{6} \left(\frac{5}{4} - (1 - \alpha)^3 - \left(\frac{\alpha}{2} \right)^3 - \left(1 - \frac{\alpha}{2} \right)^3 - \frac{3}{2} (1 - \alpha)^2 \right) \\ &> \frac{1}{6} \times \frac{4}{5}. \end{aligned} \tag{4.28}$$

Subcase 3.7. $-1 \leq z_2 \leq z_1 - 2$ and $y_2 \geq 1.2$. By the assumption $\mathbf{x}_2 = (2, y_2, z_2) \in P_2$ we get

$$2 + y_2 - z_2 < 4.$$

Thus one can deduce

$$1.2 \leq y_2 < z_2 + 2 \leq z_1,$$

which implies

$$-0.8 \leq z_2 \leq z_1 - 2$$

and

$$z_2 + 0.2 \leq y_2 + z_2 - 1 \leq 2z_2 + 1 \leq 2z_1 - 3 \leq 2\beta - 3.$$

Let T_0^\bullet denote the orthogonal tetrahedron with vertices $(1, y_2 - 1, z_2 + 1)$, $(2, y_2 - 1, z_2 + 1)$, $(1, y_2, z_2 + 1)$ and $(1, y_2 - 1, z_2)$, let P_1 be the polytope defined above (4.11), let S_8 be the halfspace defined in Subcase 2.2, and define

$$P_6 = P_1 \cap T_0^\bullet \cap S_8.$$

By Lemma 2.3 and Lemma 2.4 it can be verified that

$$P_6 \subset D'_1(C, X').$$

We recall $S_2 = \{(x, y, z) : z \geq 0\}$, $S_{10} = \{(x, y, z) : x - y - z \geq 1\}$, and $S_{11} = \{(x, y, z) : x - y + z \geq 2\}$. By routine computations we have

$$\begin{aligned} v(T_0^\bullet \cap S_{11}) &= \frac{1}{6} \times \frac{1}{4} (2 - y_2 + z_2)^3, \\ v(T_0^\bullet \cap S_2 \cap S_{10}) &\leq \frac{1}{6} \times \frac{3}{2} (y_2 + z_2 - 1)^2 (z_2 + 1), \\ v(P_6) &\geq \frac{1}{6} \left(1 - (y_2 - 1)^3 - \frac{1}{4} (2 - y_2 + z_2)^3 + z_2^3 - \frac{3}{2} (y_2 + z_2 - 1)^2 (z_2 + 1) - (1 - x_1)^3 \right) \end{aligned}$$

and therefore

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_6) \\ &\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 + (z_1 - 1)^3 - 2(1 - x_1)^3 - (y_2 - 1)^3 - \frac{1}{4} (2 - y_2 + z_2)^3 + z_2^3 \right. \\ &\quad \left. - \frac{3}{2} (y_2 + z_2 - 1)^2 (z_2 + 1) \right) \\ &\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 + (z_1 - 1)^3 - 2(1 - x_1)^3 - (z_2 + 1)^3 + z_2^3 - \frac{3}{2} (y_2 + z_2 - 1)^2 (z_2 + 1) \right). \end{aligned}$$

When $-0.8 \leq z_2 \leq -0.4$, we have

$$z_2 + 0.2 \leq y_2 + z_2 - 1 \leq 2z_2 + 1 \leq -(z_2 + 0.2)$$

and

$$\begin{aligned}
v(D'_1(C, X')) &\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 + (z_1 - 1)^3 - 2(1 - x_1)^3 - (z_2 + 1)^3 + z_2^3 - \frac{3}{2}(z_2 + 0.2)^2(z_2 + 1) \right) \\
&\geq \frac{1}{6} \left(\frac{5}{4} - (1 - \alpha)^3 - \left(\frac{1}{5}\right)^3 - \left(\frac{4}{5}\right)^3 - \frac{3}{2} \left(\frac{3}{5}\right)^2 \frac{1}{5} \right) \\
&\geq \frac{1}{6} \times \frac{3}{5}.
\end{aligned} \tag{4.29}$$

When $-0.4 \leq z_2 \leq z_1 - 2$, we get

$$-(2z_2 + 1) \leq z_2 + 0.2 \leq y_2 + z_2 - 1 \leq 2z_2 + 1$$

and

$$\begin{aligned}
v(D'_1(C, X')) &\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 + (z_1 - 1)^3 - 2(1 - x_1)^3 - (z_2 + 1)^3 + z_2^3 - \frac{3}{2}(2z_2 + 1)^2(z_2 + 1) \right) \\
&\geq \frac{1}{6} \left(1 + (y_1 - 1)^3 - 2(1 - x_1)^3 - (2 - z_1)^3 - \frac{3}{2}(2z_1 - 3)^2(z_1 - 1) \right) \\
&\geq \frac{1}{6} \left(1 + (2 - \beta)^3 - 2(1 - \alpha)^3 - (2 - \beta)^3 - \frac{3}{2}(2\beta - 3)^2(\beta - 1) \right) \\
&> \frac{1}{6} \times \frac{1}{2}.
\end{aligned} \tag{4.30}$$

As a conclusion of (4.29) and (4.30), in this subcase we have

$$v(D'_1(C, X')) > \frac{1}{6} \times \frac{1}{2}.$$

Subcase 3.8. $-1 \leq z_2 \leq z_1 - 2$ and $0 \leq y_2 \leq 1.2$. We define

$$\begin{aligned}
P_7 &= P_1 \cap \{(x, y, z) : 1 \leq x \leq x_1 + 1, \max\{0, y_2 - 1\} \leq y \leq 1, z_2 + 1 \leq z \leq z_1 - 1, \\
&\quad (x - 1) - (y - y_1 + 1) + (z - z_2 - 1) \leq 1\}, \\
P_8 &= P_1 \cap \{(x, y, z) : 1 \leq x \leq x_1 + 1, \max\{0, y_2 - 1\} \leq y \leq 1, 0 \leq z \leq z_2 + 1\} \setminus \{C + \mathbf{x}_2\}, \\
\overline{P_7} &= P_7 \cap \{(x, y, z) : z = z_2 + 1\}, \quad \widetilde{P_8} = P_8 \cap \{(x, y, z) : z = z_2 + 1\},
\end{aligned}$$

and

$$\overline{P_8} = P_8 \cap \{(x, y, z) : z = 0\}.$$

Clearly, we have

$$\text{int}(P_7) \cap \text{int}(P_8) = \emptyset.$$

By Lemmas 2.3 and 2.4 it can be shown that

$$P_i \subseteq D'_1(C, X')$$

holds for both $i = 7$ and 8 .

When $z_2 \geq y_1 - 2$, we have

$$P_7 = P_1 \cap \{(x, y, z) : 1 \leq x \leq x_1 + 1, \max\{0, y_2 - 1\} \leq y \leq 1, z_2 + 1 \leq z \leq z_1 - 1\}$$

and

$$\widetilde{P_8} \subseteq \overline{P_7}.$$

Now we consider $v(P_7)$ and $v(P_8)$ as functions of x_1, y_1, z_1, y_2 and z_2 . First, we have

$$\begin{aligned}
\frac{\partial v(P_7)}{\partial z_2} &= -s(\overline{P_7}) \leq -s(\widetilde{P_8}), \\
\frac{\partial v(P_8)}{\partial z_2} &\leq s(\overline{P_8}) \leq s(\widetilde{P_8}), \\
\frac{\partial v(P_7) + v(P_8)}{\partial z_2} &\leq -s(\widetilde{P_8}) + s(\widetilde{P_8}) \leq 0
\end{aligned}$$

and therefore

$$v(P_7) + v(P_8) \geq (v(P_7) + v(P_8)) \Big|_{z_2=z_1-2} \geq v(P_8) \Big|_{z_2=z_1-2} \geq \frac{1}{3} \left(1 - \frac{z_1}{3}\right)^3.$$

Thus, combined with (4.14), we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_7 \cup P_8) \\
&\geq \frac{1}{6} \left((y_1 - 1)^3 + (z_1 - 1)^3 - (1 - x_1)^3 + 2 \left(1 - \frac{z_1}{3} \right)^3 \right) \\
&> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right).
\end{aligned} \tag{4.31}$$

When $z_2 \leq y_1 - 2$ and $1 \leq y_2 \leq 1.2$. It can be shown that P_7 contains $(1, 1, z_2 + 1)$, $(1, y_2 - 1, z_2 + 1)$, $(1, y_2 - 1, z_1 - 1)$, $(1, 1, z_1 - 1)$ and $(2.5 + z_2, 0.5, z_2 + 1)$, and therefore

$$v(P_7) \geq \frac{1}{3}(2 - y_2)(z_1 - z_2 - 2)(1.5 + z_2).$$

If $-1 \leq z_2 \leq -0.8$, by (4.14) we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_7) \\
&\geq \frac{1}{6} ((y_1 - 1)^3 + (z_1 - 1)^3 - (1 - x_1)^3 + 2(2 - y_2)(z_1 - z_2 - 2)(1.5 + z_2)) \\
&\geq \frac{1}{6} \left(\frac{1}{4} + 2(2 - 1.2)(1.5 + 0.8 - 2)(1.5 - 1) \right) \\
&> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right).
\end{aligned} \tag{4.32}$$

If $-0.8 \leq z_2 \leq y_1 - 2 \leq -0.5$, let P_6 be the polytope defined in Subcase 3.7, by (4.14) we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_6) \\
&\geq \frac{1}{6} \left(\frac{5}{4} - (z_2 + 1)^3 + z_2^3 - \frac{3}{2} z_2^2 (z_2 + 1) - (1 - x_1)^3 \right) \\
&\geq \frac{1}{6} \left(\frac{5}{4} - \left(\frac{1}{5} \right)^3 - \left(\frac{4}{5} \right)^3 - \frac{3}{2} \left(\frac{4}{5} \right)^2 \frac{1}{5} - (1 - \alpha)^3 \right) \\
&\geq \frac{1}{6} \times \frac{1}{2}.
\end{aligned} \tag{4.33}$$

When $z_2 \leq y_1 - 2$ and $0 \leq y_2 \leq 1$. Let T_0^* to be the tetrahedron with vertices $(1, y_1 - 1, z_2 + 1)$, $(2, y_1 - 1, z_2 + 1)$, $(1, y_1 - 2, z_2 + 1)$ and $(1, y_1 - 1, z_2 + 2)$, and define

$$P_9 = P_1 \cap T_0^* \cap S_8.$$

By routine arguments it can be shown that

$$P_9 \subseteq D'_1(C, X')$$

and

$$v(P_9) \geq \frac{1}{6} \left(1 - (2 - y_1)^3 - (z_2 + 1)^3 - \frac{1}{4}(y_1 - z_2 - 2)^3 - \frac{3}{2}(1 - y_1 - z_2)^2(y_1 - 1) - (1 - x_1)^3 \right).$$

In addition, we have

$$(z_2 + 1)^3 + \frac{1}{4}(y_1 - z_2 - 2)^3 \leq (y_1 - 1)^3$$

and

$$y_1 - 2 \leq 3 - 2y_1 \leq 1 - y_1 - z_2 \leq 2 - y_1.$$

Thus, combined with (4.14), we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_9) \\
&\geq \frac{1}{6} \left(1 + (z_1 - 1)^3 - 2(1 - x_1)^3 - (2 - y_1)^3 - \frac{3}{2}(2 - y_1)^2(y_1 - 1) \right) \\
&> \frac{1}{6} \times \frac{7}{10}.
\end{aligned} \tag{4.34}$$

When $z_2 \leq y_1 - 2$ and $0 \leq y_2 \leq 1.2$, as a conclusion of (4.32), (4.33) and (4.34), we obtain

$$v(D'_1(C, X')) \geq \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \tag{4.35}$$

As a conclusion of (4.31) and (4.35), in this subcase we have

$$v(D'_1(C, X')) > \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right).$$

Subcase 3.9. $-1 \leq z_2 \leq z_1 - 2$ and $z_1 - 2 \leq y_2 \leq 0$. Let P_5 and P_9 be the polytopes defined in Subcases 3.6 and 3.8, respectively. In other words,

$$P_5 = P_1 \cap T_0^\circ \cap S_8$$

and

$$P_9 = P_1 \cap T_0^\star \cap S_8.$$

Clearly, we have

$$y_2 + z_2 > -1$$

and

$$\text{int}(P_5) \cap \text{int}(P_9) = \emptyset.$$

By Lemmas 2.3 and 2.4 it can be shown that

$$P_i \subseteq D'_1(C, X')$$

holds for both $i = 5$ and 9 .

When $-1 \leq z_2 \leq -0.8$, we have

$$z_2 \leq 1 - \beta \leq y_1 - 2, \quad y_2 \geq -1 - z_2 \geq -0.2,$$

$$v(P_9) \geq \frac{1}{6} \left(1 - (2 - y_1)^3 - (z_2 + 1)^3 - \frac{1}{4}(y_1 - z_2 - 2)^3 - \frac{3}{2}(1 - y_1 - z_2)^2(y_1 - 1) - (1 - x_1)^3 \right)$$

and therefore, by (4.14),

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_9) \\ &\geq \frac{1}{6} \left(\frac{5}{4} - (2 - y_1)^3 - \left(\frac{1}{5} \right)^3 - \frac{1}{4} \left(1 - \frac{\alpha}{2} \right)^3 - \frac{3}{2}(1 - y_1 - z_2)^2(y_1 - 1) - (1 - \alpha)^3 \right) \\ &\geq \frac{1}{6} \left(\frac{5}{4} - (\beta - 1)^3 - \left(\frac{1}{5} \right)^3 - \frac{1}{4} \left(1 - \frac{\alpha}{2} \right)^3 - \frac{3}{2}(\beta - 1)^2(2 - \beta) - (1 - \alpha)^3 \right) \\ &> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \end{aligned} \tag{4.36}$$

If $-0.8 \leq z_2 \leq z_1 - 2$ and $z_1 - 2 \leq y_2 \leq 0$, we have

$$0 \leq y_2 - z_2 \leq -z_2,$$

$$v(P_5) \geq \frac{1}{6} \left(1 + y_2^3 + z_2^3 - \frac{3}{2}(y_2 - z_2)^2(z_2 + 1) - (1 - x_1)^3 \right)$$

and

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_2 \cup T_3 \cup P_5) \\ &\geq \frac{1}{6} \left(\frac{5}{4} + y_2^3 + z_2^3 - \frac{3}{2}(y_2 - z_2)^2(z_2 + 1) - (1 - x_1)^3 \right) \\ &\geq \frac{1}{6} \left(\frac{5}{4} - \left(\frac{1}{5} \right)^3 - \left(\frac{4}{5} \right)^3 - \frac{3}{2}z_2^2(z_2 + 1) - (1 - \alpha)^3 \right) \\ &\geq \frac{1}{6} \left(\frac{5}{4} - \left(\frac{1}{5} \right)^3 - \left(\frac{4}{5} \right)^3 - \frac{3}{2} \left(\frac{2}{3} \right)^2 \frac{1}{3} - (1 - \alpha)^3 \right) \\ &> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \end{aligned} \tag{4.37}$$

By (4.36) and (4.37), in Subcase 3.9 we get

$$v(D'_1(C, X')) > \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right).$$

As a conclusion of these subcases we get

$$v(D'_1(C, X')) \geq \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right)$$

for Case 3, where the equality holds if and only if $\mathbf{x}_1 = \left(\frac{2}{3} + \frac{2}{3} \sqrt{\frac{1}{10}}, \frac{5}{3} - \frac{1}{3} \sqrt{\frac{1}{10}}, \frac{5}{3} - \frac{1}{3} \sqrt{\frac{1}{10}} \right)$ and $\mathbf{x}_2 = \left(2, \frac{1}{3} - \frac{2}{3} \sqrt{\frac{1}{10}}, \frac{1}{3} - \frac{2}{3} \sqrt{\frac{1}{10}} \right)$. Lemma 4.4 is proved. \square

Lemma 4.5. *If $\mathbf{x}_1 \in \triangle_3 \cap H$, then we have*

$$v(D_1(C, X)) \geq \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right).$$

Proof. In this case $C + \mathbf{x}_1$ touches C at an hexagon with vertices $(1, 2 - z_1, z_1 - 1)$, $(1, y_1 - 1, 2 - y_1)$, $(2 - y_1, y_1 - 1, 1)$, $(x_1 - 1, 2 - x_1, 1)$, $(x_1 - 1, 1, 2 - x_1)$ and $(2 - z_1, 1, z_1 - 1)$, as shown in Figure 3. Then, by Lemma 2.3 and Lemma 2.4 it can be shown that $D'_1(C, X')$ contains four tetrahedra T_1 , T_2 , T_3 and T_4 , where T_1 has vertices $(x_1 - 1, 1, 1)$, $(0, 1, 1)$, $(x_1 - 1, 2 - x_1, 1)$ and $(x_1 - 1, 1, 2 - x_1)$, T_2 has vertices $(1, y_1 - 1, 1)$, $(2 - y_1, y_1 - 1, 1)$, $(1, 0, 1)$ and $(1, y_1 - 1, 2 - y_1)$, T_3 has vertices $(1, 1, z_1 - 1)$, $(2 - z_1, 1, z_1 - 1)$, $(1, 2 - z_1, z_1 - 1)$ and $(1, 1, 0)$, and T_4 has vertices $(1, y_1 - 1, z_1 - 1)$, $(x_1, y_1 - 1, z_1 - 1)$, $(1, 2 - z_1, z_1 - 1)$ and $(1, y_1 - 1, 2 - y_1)$. Clearly, all T_1 , T_2 , T_3 and T_4 are homothetic to T_0 with ratios $x_1 - 1$, $y_1 - 1$, $z_1 - 1$ and $x_1 - 1$, respectively. Moreover, their interiors are pairwise disjoint. Thus, we have

$$v(T_1 \cup T_2 \cup T_3 \cup T_4) \geq \frac{1}{6} (2(x_1 - 1)^3 + (y_1 - 1)^3 + (z_1 - 1)^3) \geq \frac{1}{6} \times \frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}}, \quad (4.38)$$

where the last equality holds if and only if $\mathbf{x}_1 = \left(\frac{2\sqrt{2}+2}{2\sqrt{2}+1}, \frac{3\sqrt{2}+1}{2\sqrt{2}+1}, \frac{3\sqrt{2}+1}{2\sqrt{2}+1} \right)$. In particular, we have

$$v(T_1 \cup T_2 \cup T_3 \cup T_4) \geq \frac{1}{6} (2(x_1 - 1)^3 + (y_1 - 1)^3 + (z_1 - 1)^3) > \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right)$$

whenever $x_1 \geq 1.585$. Thus, in the rest of this proof we assume that

$$x_1 \leq 1.585. \quad (4.39)$$

When $\mathbf{x}_1 \in \triangle_3 \cap H$, one can deduce

$$J \cap (F_0 + \mathbf{x}_1) = \emptyset$$

and therefore by (4.12)

$$J \cap (\text{int}(F_0) \cup (\text{int}(F_3) + \mathbf{x}_1)) \neq \emptyset. \quad (4.40)$$

Now, we estimate $v(D'_1(C, X'))$ by considering two cases based on (4.40).

Case 1. $(\text{int}(F_3) + \mathbf{x}_1) \cap J \neq \emptyset$. Assume that $C + \mathbf{x}_2$, where $\mathbf{x}_2 = (x_2, y_2, z_2) \in X'$, touches $C + \mathbf{x}_1$ at some relative interior points of $F_3 + \mathbf{x}_1$. Clearly we have

$$y_2 \geq y_1 - 2 \quad \text{and} \quad z_2 \geq z_1 - 2 \geq y_1 - 2.$$

If $x_1 > 1.5$, by

$$\begin{cases} (x_2 - x_1) - (y_2 - y_1) - (z_2 - z_1) = 4, \\ x_1 + y_1 + z_1 = 4 \end{cases}$$

we get

$$x_2 - y_2 - z_2 = 4 + x_1 - (y_1 + z_1) = 2x_1 > 3,$$

which contradicts the assumption $\mathbf{x}_2 \in P_2$. Thus, in this case we have $x_1 \leq 1.5$ and therefore

$$z_1 \geq 2 - \frac{x_1}{2} \geq \frac{5}{4}. \quad (4.41)$$

Now we consider two subcases.

Subcase 1.1. $y_2 \geq z_1 - 2$. Let S_7 denote the halfspace $\{(x, y, z) : (x - x_1) - (y - y_1) - (z - z_1) \leq 2\}$ and define

$$P_{10} = P_1 \cap S_7.$$

By routine arguments based on Lemma 2.3 and Lemma 2.4, it can be deduced that

$$P_{10} \setminus \{C + \mathbf{x}_1\} \subset D'_1(C, X').$$

We observe that P_{10} is independent of \mathbf{x}_2 , and

$$v(P_{10} \setminus \{C + \mathbf{x}_1\}) \geq v(P_{10} \setminus \{C + \overline{\mathbf{x}}_1\}),$$

where $\overline{\mathbf{x}}_1 = (x_1, 2 - \frac{1}{2}x_1, 2 - \frac{1}{2}x_1)$. Let T_{10} denote the tetrahedron with vertices $(1, 2 - x_1, 0)$, $(x_1, 2 - x_1, 0)$, $(1, 3 - 2x_1, 0)$, and $(1, 2 - x_1, 1 - x_1)$, and let T_{11} denote the tetrahedron with vertices $(1, 0, 2 - x_1)$, $(x_1, 0, 2 - x_1)$, $(1, 1 - x_1, 2 - x_1)$ and $(1, 0, 3 - 2x_1)$. Clearly, all T_1 , T_4 , T_{10} and T_{11} are congruent to each others. For convenience, we write $\mathbf{x}_2 = (2, 1 - x_1, 1 - x_1)$, $X^* = \{\mathbf{o}, \overline{\mathbf{x}}_1, \mathbf{x}_2\}$,

$$R_5 = \{(x, y, z) \in \partial(C + \mathbf{x}_2) : x \leq 2, y \geq 1 - x_1, z \geq 1 - x_1\},$$

and define

$$P_{11} = \bigcup_{\mathbf{x} \in R_5} s(C, X^*, -\mathbf{e}_1, \mathbf{x}).$$

Then by Lemma 2.3 and Lemma 2.4 it can be verified that

$$P_{11} \subset D'_1(C, X') \cup T_{10} \cup T_{11} \setminus \{T_1 \cup T_4\}$$

and therefore

$$v(D'_1(C, X')) \geq v(P_{11}).$$

In fact, the last equality holds only if $x_1 = 1$. By reflections, it can be seen that P_{11} is certain $D'_1(C, X')$ estimated in Subcase 2.1 of Lemma 4.4. Thus, we obtain

$$v(D'_1(C, X')) > \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \quad (4.42)$$

Subcase 1.2. $y_2 \leq z_1 - 2$. Then the assumption $\text{int}(F_3 + \mathbf{x}_1) \cap J \neq \emptyset$ implies

$$y_1 - 2 \leq y_2 \leq z_1 - 2.$$

We define T'_0 to be the orthogonal tetrahedron with vertices $(1, y_2 + 1, z_1 - 1)$, $(2, y_2 + 1, z_1 - 1)$, $(1, y_2 + 2, z_1 - 1)$ and $(1, y_2 + 1, z_1 - 2)$, define S_9 to be the halfspace $\{(x, y, z) : z \geq z_2 - 1\}$, and define

$$P_3 = P_1 \cap T'_0 \cap S_9.$$

It can be verified by Lemma 2.3 and Lemma 2.4 that

$$P_3 \subseteq D'_1(C, X').$$

Similar to Subcase 2.2 of Lemma 4.4, by the assumptions on \mathbf{x}_1 and \mathbf{x}_2 (in particular (4.39)) one can deduce $z_2 \leq 2 - x_1 \leq 1$,

$$-\frac{3}{5} \leq 3 - 2\beta \leq 3 - 2z_1 \leq 1 - z_1 - y_2 \leq 3 - y_1 - z_1 = x_1 - 1 < \frac{3}{5},$$

$$v(T'_0 \setminus \{S_2 \cap S_9\}) \leq \max \left\{ \frac{1}{6}(2 - z_1)^3, \frac{1}{6}(1 - z_1 + z_2)^3 \right\} \leq \frac{1}{6}(2 - z_1)^3,$$

$$v(T'_0 \cap S_{10}) \leq \frac{1}{2} \left(\frac{1 - z_1 - y_2}{\sqrt{2}} \right)^2 (z_1 - 1) \leq \frac{1}{6} \times \frac{3}{2} \left(\frac{3}{5} \right)^2 (z_1 - 1),$$

$$v(T'_0 \cap S_{11}) \leq \frac{1}{6} \times 2 \left(\frac{z_1 - y_2 - 2}{2} \right)^3 \leq \frac{1}{6} \times \frac{1}{4} (z_1 - y_1)^3,$$

and therefore

$$v(P_3) \geq \frac{1}{6} \left(1 - (1 + y_2)^3 - (2 - z_1)^3 - \frac{3}{2} \left(\frac{3}{5} \right)^2 (z_1 - 1) - \frac{1}{4} (z_1 - y_1)^3 \right).$$

Thus, by (4.38), (4.41) and routine computations we get

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_3) \\ &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 - (2 - z_1)^3 - \frac{3}{2} \left(\frac{3}{5} \right)^2 (z_1 - 1) - \frac{1}{4} (z_1 - y_1)^3 \right) \\ &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 - (2 - z_1)^3 - \frac{3}{2} \left(\frac{3}{5} \right)^2 (z_1 - 1) - \frac{1}{4} (z_1 - 1)^3 \right) \\ &\geq \frac{1}{6} \left(1 - (2 - \beta)^3 - \frac{3}{2} \left(\frac{3}{5} \right)^2 (\beta - 1) - \frac{1}{4} (\beta - 1)^3 \right) \\ &> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \end{aligned} \quad (4.43)$$

As a conclusion of (4.42) and (4.43), in this case we obtain

$$v(D'_1(C, X')) > \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right).$$

Case 2. $\text{int}(F_0) \cap J \neq \emptyset$. Assume that $C + \mathbf{x}_2$, where $\mathbf{x}_2 = (2, y_2, z_2) \in X'$, touches C at some interior points of F_0 . Similar to Case 3 of Lemma 4.4, we divide the region $\{(y_2, z_2) : -1 \leq y_2 \leq 2, -1 \leq z_2 \leq 2\}$ into ten parts as illustrated in Figure 8 and consider the corresponding subcases.

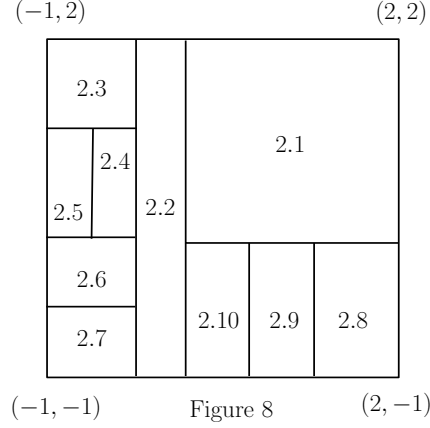


Figure 8

Subcase 2.1. $y_2 \geq z_1 - 2$ and $z_2 \geq z_1 - 2$. Then, it can be deduced that the hyperplane $(x - x_1) - (y - y_1) - (z - z_1) = 2$ separates $C + \mathbf{x}_1$ and $C + \mathbf{x}_2$. Therefore, similar to Case 1, we have $x_1 \leq \frac{3}{2}$. Let S_7 denote the halfspace $\{(x, y, z) : (x - x_1) - (y - y_1) - (z - z_1) \leq 2\}$ and define

$$P_{10} = P_1 \cap S_7.$$

By routine arguments based on Lemma 2.3 and Lemma 2.4, it can be deduced that

$$P_{10} \setminus \{C + \mathbf{x}_1\} \subseteq D'_1(C, X').$$

It is easy to see that P_{10} is independent of \mathbf{x}_2 , up to \mathbf{x}_2 satisfying $x_2 = 2$, $y_2 \geq z_1 - 2$ and $z_2 \geq z_1 - 2$, and

$$v(P_{10} \setminus \{C + \mathbf{x}_1\}) \geq v(P_{10} \setminus \{C + \overline{\mathbf{x}}_1\}),$$

where $\overline{\mathbf{x}}_1 = (x_1, 2 - \frac{1}{2}x_1, 2 - \frac{1}{2}x_1)$. Then, by arguments similar to Subcase 1.1, it can be proved that

$$v(D'_1(C, X')) > \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \quad (4.44)$$

Subcase 2.2. $y_1 - 2 \leq y_2 \leq z_1 - 2$. Let P_3 be the polytope defined in Subcase 1.2, here we also have

$$P_3 \subseteq D'_1(C, X').$$

In fact, in this case we have either

$$z_1 - z_2 \geq 2$$

or

$$(2 - x_1) - (y_2 + 1 - y_1) - (z_2 + 1 - z_1) \geq 2,$$

which implies

$$z_1 - z_2 \geq 2 + (y_2 - y_1) + x_1 \geq x_1 \geq 1.$$

Then we have

$$v(P_3) \geq \frac{1}{6} \left(1 - (1 + y_2)^3 - (2 - z_1)^3 - \frac{3}{2} \left(\frac{3}{5} \right)^2 (z_1 - 1) - \frac{1}{4} (z_1 - y_1)^3 \right).$$

On the other hand, by the assumption $\mathbf{x}_1 \in \Delta_3 \cap H$ and (4.39) we get

$$1.2075 \leq z_1 \leq \beta.$$

Thus, by (4.38) and routine computations we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_3) \\
&\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 - (2 - z_1)^3 - \frac{3}{2} \left(\frac{3}{5} \right)^2 (z_1 - 1) - \frac{1}{4} (z_1 - y_1)^3 \right) \\
&\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 - (2 - z_1)^3 - \frac{3}{2} \left(\frac{3}{5} \right)^2 (z_1 - 1) - \frac{1}{4} (z_1 - 1)^3 \right) \\
&\geq \frac{1}{6} \left(1 - (2 - \beta)^3 - \frac{3}{2} \left(\frac{3}{5} \right)^2 (\beta - 1) - \frac{1}{4} (\beta - 1)^3 \right) \\
&> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \tag{4.45}
\end{aligned}$$

Subcase 2.3. $-1 \leq y_2 \leq y_1 - 2$ and $z_2 \geq 1.16$. Then, by $\mathbf{x}_1 \in \Delta_3 \cap H$ and

$$2 - y_2 + z_2 \leq 4$$

one can deduce

$$\begin{aligned}
1.16 &\leq z_2 \leq y_2 + 2 \leq y_1, \\
-0.84 &\leq y_2 \leq y_1 - 2 \leq -0.5,
\end{aligned}$$

and

$$-0.68 \leq y_2 + z_2 - 1 \leq 2y_1 - 3 \leq 0.$$

Let T_0^* denote the orthogonal tetrahedron with vertices $(1, y_2 + 1, z_2 - 1)$, $(2, y_2 + 1, z_2 - 1)$, $(1, y_2, z_2 - 1)$ and $(1, y_2 + 1, z_2)$, and let P_4 be the polytope defined in Subcase 3.3 of Lemma 4.4. It can be verified by Lemma 2.3 and Lemma 2.4 that

$$P_4 \subset D'_1(C, X').$$

Recall that $S_1 = \{(x, y, z) : y \geq 0\}$, $S'_6 = \{(x, y, z) : x + y - z \geq 2\}$ and $S_{10} = \{(x, y, z) : x - y - z \geq 1\}$, we have

$$\begin{aligned}
v(T_0^* \cap S'_6) &= 2 \times \frac{1}{6} \left(\frac{2 + y_2 - z_2}{2} \right)^3 = \frac{1}{6} \times \frac{1}{4} (2 + y_2 - z_2)^3, \\
v(T_0^* \cap S_1 \cap S_{10}) &\leq \frac{1}{6} \times \frac{3}{2} (y_2 + z_2 - 1)^2 (y_2 + 1)
\end{aligned}$$

and therefore, based on the assumptions on \mathbf{x}_1 and \mathbf{x}_2 ,

$$\begin{aligned}
v(P_4) &\geq \frac{1}{6} \left(1 - (z_2 - 1)^3 - \frac{1}{4} (2 + y_2 - z_2)^3 + y_2^3 - \frac{3}{2} (y_2 + z_2 - 1)^2 (y_2 + 1) \right) \\
&\geq \frac{1}{6} \left(1 - (y_2 + 1)^3 + y_2^3 - \frac{3}{2} (y_2 + z_2 - 1)^2 (y_2 + 1) \right) \\
&\geq \frac{1}{6} \left(1 - (y_2 + 1)^3 + y_2^3 - \frac{3}{2} \times 0.68^2 (y_2 + 1) \right) \\
&\geq \frac{1}{6} \left(1 - (1 - 0.84)^3 - 0.84^3 - \frac{3}{2} \times 0.68^2 (1 - 0.84) \right).
\end{aligned}$$

Then, combined with (4.38), we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_4) \\
&\geq \frac{1}{6} \left(1 + \frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}} - (1 - 0.84)^3 - 0.84^3 - \frac{3}{2} \times 0.68^2 (1 - 0.84) \right) \\
&> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right). \tag{4.46}
\end{aligned}$$

Subcase 2.4. $-0.8 \leq y_2 \leq y_1 - 2$ and $1 \leq z_2 \leq 1.16$. These conditions and $\mathbf{x}_1 \in \Delta_3 \cap H$ implies

$$-0.8 \leq y_2 \leq -0.5.$$

Let P_4 be the polytope defined in Subcase 3.3 of Lemma 4.4. Then we have

$$P_4 \subset D'_1(C, X')$$

and

$$\begin{aligned}
v(P_4) &\geq \frac{1}{6} \left(1 - (z_2 - 1)^3 - \frac{1}{4}(2 + y_2 - z_2)^3 + y_2^3 - \frac{3}{2}(y_2 + z_2 - 1)^2(y_2 + 1) \right) \\
&\geq \frac{1}{6} \left(1 - (1.16 - 1)^3 - \frac{1}{4}(1 + y_2)^3 + y_2^3 - \frac{3}{2}y_2^2(1 + y_2) \right) \\
&\geq \frac{1}{6} \left(1 - 0.16^3 - \frac{1}{4}(1 - 0.8)^3 - 0.8^3 - \frac{3}{2} \times 0.8^2(1 - 0.8) \right).
\end{aligned}$$

Then, combined with (4.38), we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_4) \\
&\geq \frac{1}{6} \left(1 + \frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}} - 0.16^3 - \frac{1}{4}(1 - 0.8)^3 - 0.8^3 - \frac{3}{2} \times 0.8^2(1 - 0.8) \right) \\
&> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9}\sqrt{\frac{1}{10}} \right).
\end{aligned} \tag{4.47}$$

Subcase 2.5. $-1 \leq y_2 \leq \min\{-0.8, y_1 - 2\}$ and $1 \leq z_2 \leq 1.16$. Then, by the fact that $(2, y_2, z_2) \in P_2$ one can deduce

$$y_2 + 1 \geq z_2 - 1. \tag{4.48}$$

Let P_3 and P_4 be the polytopes defined in Subcases 2.2 and 3.3 of Lemma 4.4, respectively. It can be verified that

$$\text{int}(P_3) \cap \text{int}(P_4) = \emptyset$$

and, by Lemma 2.3 and Lemma 2.4,

$$P_i \subset D'_1(C, X'), \quad i = 3, 4.$$

It can be deduced that

$$v(P_3) \geq \frac{1}{6} \left(1 - (y_2 + 1)^3 - \frac{1}{4}(z_1 - y_2 - 2)^3 - (1 - z_1 + z_2)^3 - \frac{3}{2}(1 - y_2 - z_1)^2(z_1 - z_2) \right).$$

In particular, when $1.35 \leq z_1 \leq \beta$, $-1 \leq y_2 \leq -0.8$ and $1 \leq z_2 \leq 1.16$, by routine computations we get

$$\begin{aligned}
v(P_3) &\geq \frac{1}{6} \left(1 - 0.2^3 - \frac{1}{4}(z_1 - 1)^3 - (1 - z_1 + z_2)^3 - \frac{3}{2}(2 - z_1)^2(z_1 - z_2) \right) \\
&\geq \frac{1}{6} \left(1 - 0.2^3 - \frac{1}{4}(z_1 - 1)^3 - (2.16 - z_1)^3 - \frac{3}{2}(2 - z_1)^2(z_1 - 1.16) \right) \\
&\geq \frac{1}{6} \left(1 - 0.2^3 - \frac{1}{4}(z_1 - 1)^3 - (2.16 - z_1)^3 - \frac{3}{2} \times 0.65^2(z_1 - 1.16) \right) \\
&> \frac{1}{6} \times \frac{3}{10}
\end{aligned}$$

and therefore

$$v(D'_1(C, X')) \geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_3) \geq \frac{1}{6} \left(\frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}} + \frac{3}{10} \right) > \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9}\sqrt{\frac{1}{10}} \right). \tag{4.49}$$

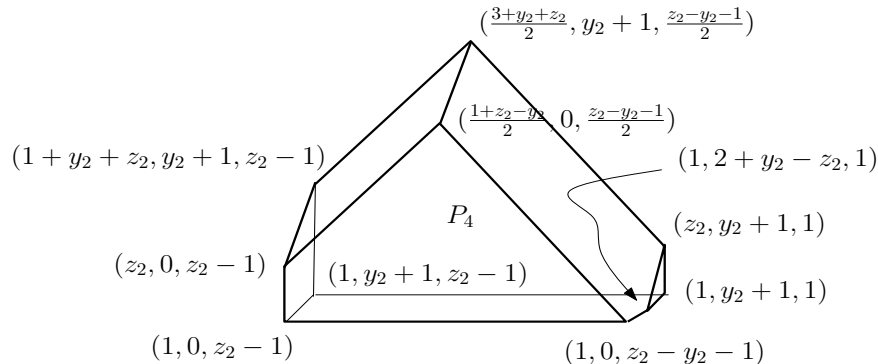


Figure 9

In fact, as shown in Figure 9, P_4 is the polytope with ten vertices $(1, y_2+1, z_2-1)$, $(1+y_2+z_2, y_2+1, z_2-1)$, $(\frac{3+y_2+z_2}{2}, y_2+1, \frac{z_2-y_2-1}{2})$, $(z_2, y_2+1, 1)$, $(1, y_2+1, 1)$, $(1, 0, z_2-1)$, $(z_2, 0, z_2-1)$, $(\frac{1+z_2-y_2}{2}, 0, \frac{z_2-y_2-1}{2})$, $(1, 0, z_2-y_2-1)$ and $(1, 2+y_2-z_2, 1)$. We observe that P_4 can be obtained by cutting of two tetrahedra and one prismoid from the tetrahedron with vertices $(\frac{3+y_2+z_2}{2}, y_2+1, \frac{z_2-y_2-1}{2})$, $(1, y_2+1, -y_2-1)$, $(1, y_2+1, z_2)$ and $(1, \frac{y_2-z_2+1}{2}, \frac{z_2-y_2-1}{2})$, where the last tetrahedron is the union of two tetrahedra, both of them are homothetic to T_0 with ratio $\frac{y_2+z_2+1}{2}$. Then, by (4.48), $-1 \leq y_2 \leq -0.8$ and $1 \leq z_2 \leq 1.16$, we get

$$\begin{aligned}
v(P_4) &= \frac{1}{6} \left(\frac{1}{4}(z_2 + (y_2 + 1))^3 - \frac{1}{4}(z_2 - (y_2 + 1))^3 - (z_2 - 1)^3 - (y_2 + 1)((y_2 + z_2)^2 \right. \\
&\quad \left. + (z_2 - 1)^2 + (y_2 + z_2)(z_2 - 1)) \right) \\
&\geq \frac{1}{6} \left(\frac{3}{2}z_2^2(y_2 + 1) + \frac{1}{2}(y_2 + 1)^3 - (y_2 + 1)(z_2 - 1)^2 - (y_2 + 1)((y_2 + z_2)^2 \right. \\
&\quad \left. + (z_2 - 1)^2 + (y_2 + z_2)(z_2 - 1)) \right) \\
&\geq \frac{y_2 + 1}{6} \left(\frac{3}{2}z_2^2 + \frac{1}{2}(1 - 0.8)^2 - 2(z_2 - 1)^2 - 2(z_2 - 0.8)^2 \right) \\
&\geq \frac{1}{6} \times 1.4 \times (y_2 + 1).
\end{aligned} \tag{4.50}$$

Similarly, when $1.2075 \leq z_1 \leq 1.35$, noticing $y_2 + 1 + z_1 = (y_2 + 1 + z_2) + (z_1 - z_2)$ and $y_2 + 1 - z_1 + 2z_2 = (y_2 + 1 + z_2) - (z_1 - z_2)$, we get

$$\begin{aligned}
v(P_3) &\geq \frac{1}{6} \left(\frac{1}{4}(y_2 + 1 + z_1)^3 - \frac{1}{4}(y_2 + 1 - z_1 + 2z_2)^3 - 3(z_1 - z_2)((y_2 + z_1)^2 + (y_2 + 1)^2) \right) \\
&\geq \frac{z_1 - z_2}{6} \left(\frac{3}{2}(y_2 + 1 + z_2)^2 + \frac{1}{2}(z_1 - z_2)^2 - 3(y_2 + z_1)^2 - 3(y_2 + 1)^2 \right) \\
&\geq \frac{1}{6} \left(\frac{3}{2} - \frac{5}{2}(z_1 - 1)^2 \right) (z_1 - z_2).
\end{aligned} \tag{4.51}$$

At the same time, we have

$$\frac{3}{2} - \frac{5}{2}(z_1 - 1)^2 \leq 1.4. \tag{4.52}$$

Then, by (4.38), (4.50)-(4.52) and (4.48) we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_3 \cup P_4) \\
&\geq \frac{1}{6} \left(\frac{4\sqrt{2}+2}{25+22\sqrt{2}} + \left(\frac{3}{2} - \frac{5}{2}(z_1 - 1)^2 \right) (z_1 - z_2) + 1.4(y_2 + 1) \right) \\
&\geq \frac{1}{6} \left(\frac{4\sqrt{2}+2}{25+22\sqrt{2}} + \left(\frac{3}{2} - \frac{5}{2}(z_1 - 1)^2 \right) (z_1 - z_2 + y_2 + 1) \right) \\
&\geq \frac{1}{6} \left(\frac{4\sqrt{2}+2}{25+22\sqrt{2}} + \left(\frac{3}{2} - \frac{5}{2}(z_1 - 1)^2 \right) (z_1 - 1) \right) \\
&> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9}\sqrt{\frac{1}{10}} \right).
\end{aligned} \tag{4.53}$$

Subcase 2.6. $-1 \leq y_2 \leq y_1 - 2$ and $y_1 - 2 \leq z_2 \leq 1$. Then we get

$$P_3 \subset D'_1(C, X'),$$

where P_3 was defined in Subcase 2.2 of Lemma 4.4, and

$$v(P_3) \geq \frac{1}{6} \left(1 - (1 + y_2)^3 - \frac{1}{4}(z_1 - y_2 - 2)^3 - (2 - z_1)^3 - \frac{3}{2}(1 - y_2 - z_1)^2(z_1 - 1) \right).$$

By routine computations one can deduce

$$\begin{aligned}
(1 + y_2)^3 + \frac{1}{4}(z_1 - y_2 - 2)^3 &\leq (z_1 - 1)^3, \\
x_1 - 1 = 3 - y_1 - z_1 &\leq 1 - y_2 - z_1 \leq 2 - z_1
\end{aligned}$$

and

$$(1 - y_2 - z_1)^2(z_1 - 1) \leq (2 - z_1)^2(z_1 - 1).$$

On the other hand, when $\mathbf{x}_1 \in \Delta_3 \cap H$, by (4.39) we get

$$z_1 \geq 2 - \frac{x_1}{2} \geq 1.2075.$$

Thus, by the assumptions on \mathbf{x}_1 , \mathbf{x}_2 and (4.38) we get

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_3) \\ &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 + (z_1 - 1)^3 - (1 + y_2)^3 - \frac{1}{4}(z_1 - y_2 - 2)^3 \right. \\ &\quad \left. - (2 - z_1)^3 - \frac{3}{2}(1 - y_2 - z_1)^2(z_1 - 1) \right) \\ &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 - (2 - z_1)^3 - \frac{3}{2}(2 - z_1)^2(z_1 - 1) \right) \\ &= \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 - \frac{1}{2}(2 - z_1)^2(1 + z_1) \right) \\ &\geq \frac{1}{6} \left(1 + \frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}} - (z_1 - 1)^3 - \frac{1}{2}(2 - z_1)^2(1 + z_1) \right) \\ &\geq \frac{1}{6} \left(1 + \frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}} - (1.2075 - 1)^3 - \frac{1}{2}(2 - 1.2075)^2(1 + 1.2075) \right) \\ &> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9}\sqrt{\frac{1}{10}} \right). \end{aligned} \tag{4.54}$$

Subcase 2.7. $-1 \leq y_2 \leq y_1 - 2$ and $z_2 < y_1 - 2$. First, by the assumption $\mathbf{x}_2 = (2, y_2, z_2) \in P_2$ one can deduce

$$2 - y_2 - z_2 < 3$$

and therefore

$$y_2 + z_2 > -1. \tag{4.55}$$

On the other hand, by the assumption $y_2 \leq y_1 - 2$, $z_2 < y_1 - 2$ and $\mathbf{x}_1 \in \Delta_3 \cap H$ we get

$$y_2 + z_2 < 2y_1 - 4 \leq -1,$$

which contradicts (4.55). Therefore this subcase can't happen.

Subcase 2.8. $-1 \leq z_2 \leq z_1 - 2$ and $y_2 \geq 1.2$. By the assumption $\mathbf{x}_2 = (2, y_2, z_2) \in P_2$ we get

$$2 + y_2 - z_2 < 4.$$

Thus one can deduce

$$1.2 \leq y_2 < z_2 + 2 \leq z_1,$$

which implies

$$-0.8 \leq z_2 \leq z_1 - 2$$

and

$$z_2 + 0.2 \leq y_2 + z_2 - 1 \leq 2z_2 + 1 \leq 2z_1 - 3 \leq 2\beta - 3.$$

Let T_0^\bullet denote the orthogonal tetrahedron with vertices $(1, y_2 - 1, z_2 + 1)$, $(2, y_2 - 1, z_2 + 1)$, $(1, y_2, z_2 + 1)$ and $(1, y_2 - 1, z_2)$, and let P_6 be the polytope defined in Subcase 3.7 of Lemma 4.4. That is

$$P_6 = P_1 \cap T_0^\bullet.$$

By Lemma 2.3 and Lemma 2.4 it can be verified that

$$P_6 \subset D'_1(C, X').$$

We recall $S_2 = \{(x, y, z) : z \geq 0\}$, $S_{10} = \{(x, y, z) : x - y - z \geq 1\}$, and $S_{11} = \{(x, y, z) : x - y + z \geq 2\}$. By routine computations we get

$$\begin{aligned} v(T_0^\bullet \cap S_{11}) &= \frac{1}{6} \times \frac{1}{4}(2 - y_2 + z_2)^3, \\ v(T_0^\bullet \cap S_2 \cap S_{10}) &\leq \frac{1}{6} \times \frac{3}{2}(y_2 + z_2 - 1)^2(z_2 + 1), \end{aligned}$$

$$v(P_6) \geq \frac{1}{6} \left(1 - (y_2 - 1)^3 - \frac{1}{4}(2 - y_2 + z_2)^3 + z_2^3 - \frac{3}{2}(y_2 + z_2 - 1)^2(z_2 + 1) \right)$$

and therefore

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_6) \\ &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 + (z_1 - 1)^3 - (y_2 - 1)^3 - \frac{1}{4}(2 - y_2 + z_2)^3 + z_2^3 \right. \\ &\quad \left. - \frac{3}{2}(y_2 + z_2 - 1)^2(z_2 + 1) \right) \\ &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 + (z_1 - 1)^3 - (z_2 + 1)^3 + z_2^3 - \frac{3}{2}(y_2 + z_2 - 1)^2(z_2 + 1) \right). \end{aligned}$$

When $-0.8 \leq z_2 \leq -0.4$, we have

$$z_2 + 0.2 \leq y_2 + z_2 - 1 \leq 2z_2 + 1 \leq -(z_2 + 0.2)$$

and

$$\begin{aligned} v(D'_1(C, X')) &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 + (z_1 - 1)^3 - (z_2 + 1)^3 + z_2^3 - \frac{3}{2}(z_2 + 0.2)^2(z_2 + 1) \right) \\ &\geq \frac{1}{6} \left(1 + \frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}} - \left(\frac{1}{5}\right)^3 - \left(\frac{4}{5}\right)^3 - \frac{3}{2} \left(\frac{3}{5}\right)^2 \frac{1}{5} \right) \\ &> \frac{1}{6} \times \frac{1}{2}. \end{aligned} \tag{4.56}$$

When $-0.4 \leq z_2 \leq z_1 - 2$, we get

$$-(2z_2 + 1) \leq z_2 + 0.2 \leq y_2 + z_2 - 1 \leq 2z_2 + 1$$

and

$$\begin{aligned} v(D'_1(C, X')) &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 + (z_1 - 1)^3 - (z_2 + 1)^3 + z_2^3 - \frac{3}{2}(2z_2 + 1)^2(z_2 + 1) \right) \\ &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 - (2 - z_1)^3 - \frac{3}{2}(2z_1 - 3)^2(z_1 - 1) \right) \\ &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 - (2 - z_1)^3 - \frac{3}{2}(2z_1 - 3)^2(z_1 - 1) \right) \\ &\geq \frac{1}{6} \left(1 - (2 - \beta)^3 - \frac{3}{2}(2\beta - 3)^2(\beta - 1) \right) \\ &> \frac{1}{6} \times \frac{1}{2}. \end{aligned} \tag{4.57}$$

As a conclusion of (4.56) and (4.57), in this subcase we have

$$v(D'_1(C, X')) > \frac{1}{6} \times \frac{1}{2}.$$

Subcase 2.9. $-1 \leq z_2 \leq z_1 - 2$ and $1 \leq y_2 \leq 1.2$. Let P_6 denote the polytope defined in Subcase 3.7 of Lemma 4.4 and define

$$\begin{aligned} P_7 &= P_1 \cap \{(x, y, z) : 1 \leq x \leq 2, y_2 - 1 \leq y \leq 1, z_2 + 1 \leq z \leq z_1 - 1, \\ &\quad (x - 1) - (y - y_1 + 1) + (z - z_2 - 1) \leq 1\}. \end{aligned}$$

Clearly, we have

$$\text{int}(P_6) \cap \text{int}(P_7) = \emptyset.$$

By Lemmas 2.3 and 2.4 it can be shown that

$$P_i \subseteq D'_1(C, X')$$

holds for both $i = 6$ and 7 .

For convenience, we write $\gamma(z_1) = \min\{1, z_1 - 0.5, 2.5 - z_1\}$. It can be verified that $\gamma(z_1) \geq 0.7$ and P_7 contains $(1, 1, z_2 + 1)$, $(1, y_2 - 1, z_2 + 1)$, $(1, y_2 - 1, z_1 - 1)$, $(1, 1, z_1 - 1)$, $(1.5, 0.5, z_2 + 1)$ and $(1 + \gamma(z_1), 0.5, z_1 - 1)$, and therefore

$$\begin{aligned} v(P_7) &\geq \frac{1}{3}(2 - y_2)(z_1 - z_2 - 2)\gamma(z_1) + \frac{1}{6}(2 - y_2)(z_1 - z_2 - 2)(1.5 - 1) \\ &\geq \frac{1}{6} \times 1.52(z_1 - z_2 - 2). \end{aligned}$$

In addition, by $2 + y_2 - z_2 \leq 4$ we get

$$\begin{aligned} v(P_6) &\geq \frac{1}{6} \left(1 - (y_2 - 1)^3 - \frac{1}{4}(2 - y_2 + z_2)^3 + z_2^3 - \frac{3}{2}(y_2 + z_2 - 1)^2(z_2 + 1) \right) \\ &\geq \frac{1}{6} \left(1 - (z_2 + 1)^3 + z_2^3 - \frac{3}{2}z_2^2(z_2 + 1) \right). \end{aligned}$$

Then we have

$$\begin{aligned} v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_6 \cup P_7) \\ &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 + (z_1 - 1)^3 + 1.52(z_1 - z_2 - 2) - (z_2 + 1)^3 + z_2^3 \right. \\ &\quad \left. - \frac{3}{2}z_2^2(z_2 + 1) \right) \\ &\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (y_1 - 1)^3 - (2 - z_1)^3 - \frac{3}{2}(2 - z_1)^2(z_1 - 1) \right) \\ &\geq \frac{1}{6} \left(1 + \frac{4\sqrt{2} + 2}{25 + 22\sqrt{2}} - (z_1 - 1)^3 - (2 - z_1)^3 - \frac{3}{2}(2 - z_1)^2(z_1 - 1) \right) \\ &> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9}\sqrt{\frac{1}{10}} \right). \end{aligned} \tag{4.58}$$

Subcase 2.10. $-1 \leq z_2 \leq z_1 - 2$ and $z_1 - 2 \leq y_2 \leq 1$. Then, by $y_2 + z_2 > -1$ we get $-0.5 \leq y_2$. We define

$$\begin{aligned} P_7 &= P_1 \cap \{(x, y, z) : 1 \leq x \leq 2, 0 \leq y \leq 1, z_2 + 1 \leq z \leq z_1 - 1, \\ &\quad (x - 1) - (y - y_1 + 1) + (z - z_2 - 1) \leq 1\}, \\ P_8 &= P_1 \cap \{(x, y, z) : 1 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq z_2 + 1\} \setminus \{C + \mathbf{x}_2\}, \\ \overline{P_7} &= P_7 \cap \{(x, y, z) : z = z_2 + 1\}, \quad \widetilde{P_8} = P_8 \cap \{(x, y, z) : z = z_2 + 1\}, \end{aligned}$$

and

$$\overline{P_8} = P_8 \cap \{(x, y, z) : z = 0\}.$$

In addition, we define

$$\begin{aligned} P_{12} &= P_1 \cap \{(x, y, z) : 1 \leq x \leq 2, 0 \leq y \leq y_1 - 1, z_1 - 1 \leq z \leq 1, \\ &\quad (x - 1) - (y - y_1 + 1) + (z - z_2 - 1) \leq 1\}. \end{aligned}$$

Clearly, $\text{int}(P_7)$, $\text{int}(P_8)$ and $\text{int}(P_{12})$ are pairwise disjoint. By Lemmas 2.3 and 2.4 it can be shown that

$$P_i \subseteq D'_1(C, X')$$

holds for all $i = 7, 8$ and 12 .

When $z_2 \geq y_1 - 2$, we have

$$P_7 = P_1 \cap \{(x, y, z) : 1 \leq x \leq 2, 0 \leq y \leq 1, z_2 + 1 \leq z \leq z_1 - 1\}$$

and

$$P_{12} = P_1 \cap \{(x, y, z) : 1 \leq x \leq 2, 0 \leq y \leq y_1 - 1, z_1 - 1 \leq z \leq 1\}.$$

Now we consider $v(P_7)$, $v(P_8)$ and $v(P_{12})$ as functions of x_1 , y_1 , z_1 , y_2 and z_2 . First, we have

$$\begin{aligned} \frac{\partial v(P_7)}{\partial z_2} &= -s(\overline{P_7}) \leq -s(\widetilde{P_8}), \\ \frac{\partial v(P_8)}{\partial z_2} &\leq s(\overline{P_8}) \leq s(\widetilde{P_8}), \\ \frac{\partial v(P_7) + v(P_8)}{\partial z_2} &\leq -s(\widetilde{P_8}) + s(\widetilde{P_8}) \leq 0 \end{aligned}$$

and therefore

$$\begin{aligned}
v(P_7) + v(P_8) &\geq (v(P_7) + v(P_8)) \Big|_{z_2=z_1-2} \geq v(P_8) \Big|_{z_2=z_1-2} \\
&\geq \begin{cases} \frac{1}{6} \left(\frac{1}{2} (z_1 - \frac{1}{2})^3 - 2(z_1 - 1)^3 - \frac{1}{2} (\frac{3}{2} - z_1)^3 \right), & \text{if } z_1 \leq \frac{3}{2}, \\ \frac{1}{3} \left(1 - \frac{z_1}{3} \right)^3, & \text{if } z_1 \geq \frac{3}{2}, \end{cases} \\
&= \begin{cases} \frac{1}{6} \left(3 \left(\frac{1}{2} \right)^2 (z_1 - 1) - (z_1 - 1)^3 \right), & \text{if } z_1 \leq \frac{3}{2}, \\ \frac{1}{3} \left(1 - \frac{z_1}{3} \right)^3, & \text{if } z_1 \geq \frac{3}{2}. \end{cases}
\end{aligned}$$

At the same time, we have

$$v(P_{12}) = \begin{cases} \left(\left(\frac{1}{2} \right)^3 + \frac{z_1 - 0.5}{2} \left(\frac{3}{2} - z_1 \right) \right) (y_1 - 1) + \frac{1}{2} (y_1 - 1)^2 (2 - z_1), & \text{if } z_1 \leq \frac{3}{2}, \\ \frac{1}{2} (2 - z_1)^2 (y_1 - 1) + \frac{1}{2} (y_1 - 1)^2 (2 - z_1), & \text{if } z_1 \geq \frac{3}{2}. \end{cases}$$

For example, when $y_2 = \frac{1}{2}$ and $z_2 \leq -\frac{1}{2}$, the polytope P_8 can be illustrated by Figure 10.

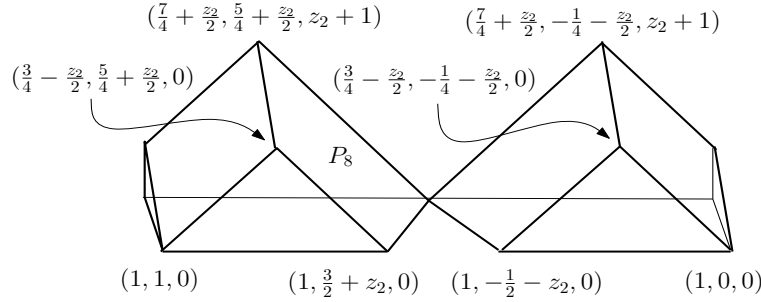


Figure 10

When $z_1 \leq \frac{3}{2}$, combined with (4.38), we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_7 \cup P_8 \cup P_{12}) \\
&\geq \frac{1}{6} \left(2(x_1 - 1)^3 + (y_1 - 1)^3 + \frac{3}{4}(y_1 - 1) + \frac{3}{4}(z_1 - 1) \right) \\
&= \frac{1}{6} \left(2(x_1 - 1)^3 + (y_1 - 1)^3 + \frac{3}{4}(2 - x_1) \right) \\
&> \frac{1}{6} \times \frac{1}{2}.
\end{aligned} \tag{4.59}$$

When $z_1 \geq \frac{3}{2}$, combined with (4.38), we get

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_7 \cup P_8 \cup P_{12}) \\
&\geq \frac{1}{6} \left(2(x_1 - 1)^3 + (y_1 - 1)^3 + (z_1 - 1)^3 + 2 \left(1 - \frac{z_1}{3} \right)^3 \right. \\
&\quad \left. + 3(2 - z_1)^2 (y_1 - 1) + 3(y_1 - 1)^2 (2 - z_1) \right) \\
&\geq \frac{1}{6} \left(2(x_1 - 1)^3 + 3(2 - z_1)^2 (y_1 - 1) + (z_1 - 1)^3 + 2 \left(1 - \frac{z_1}{3} \right)^3 \right) \\
&> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9} \sqrt{\frac{1}{10}} \right).
\end{aligned} \tag{4.60}$$

When $z_2 \leq y_1 - 2 \leq -0.5$, let P_9 be the polytope defined in Subcase 3.8 of Lemma 4.4 and define

$$P_{13} = P_1 \cap \{(x, y, z) : 1 \leq x \leq 2, y_1 - 1 \leq y \leq 1, z_2 + 1 \leq z \leq z_1 - 1\}.$$

First, it can be verified that $\text{int}(P_8)$, $\text{int}(P_9)$ and $\text{int}(P_{13})$ are pairwise disjoint, and

$$P_i \subseteq D'_1(C, X'), \quad i = 8, 9, 13.$$

Then, by determining the vertices of P_8 and P_{13} and routine computations one can deduce

$$\begin{aligned}
v(P_8) &\geq v(P_8) \Big|_{y_2=\frac{1}{2}} = \frac{1}{6} \left(\frac{1}{2} \left(z_2 + \frac{3}{2} \right)^3 - 2(z_2 + 1)^3 + \frac{1}{2} \left(z_2 + \frac{1}{2} \right)^3 \right) \\
&= \frac{1}{6} \left(\frac{3}{4}(z_2 + 1) - (z_2 + 1)^3 \right), \\
v(P_9) &\geq \frac{1}{6} \left(1 - (2 - y_1)^3 - (z_2 + 1)^3 - \frac{1}{4}(y_1 - z_2 - 2)^3 - \frac{3}{2}(1 - y_1 - z_2)^2(y_1 - 1) \right) \\
&\geq \frac{1}{6} \left(1 - (2 - y_1)^3 - (y_1 - 1)^3 - \frac{3}{2}(2 - y_1)^2(y_1 - 1) \right), \\
v(P_{13}) &\geq \left(\frac{4z_2 + 5}{8} + \frac{4z_2 + y_1 + 3}{4} \left(\frac{3}{2} - y_1 \right) \right) (z_1 - z_2 - 2) \geq \frac{4z_2 + 5}{8} (z_1 - z_2 - 2),
\end{aligned}$$

where P_{13} can be illustrated by a figure similar to Figure 9, and therefore

$$\begin{aligned}
v(D'_1(C, X')) &\geq v(T_1 \cup T_2 \cup T_3 \cup T_4 \cup P_8 \cup P_9 \cup P_{13}) \\
&\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (z_1 - 1)^3 - (2 - y_1)^3 - \frac{3}{2}(2 - y_1)^2(y_1 - 1) \right. \\
&\quad \left. + \frac{3}{4}(z_2 + 1) - (z_2 - 1)^3 + \frac{3}{4}(4z_2 + 5)(z_1 - z_2 - 2) \right). \\
&\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + (z_1 - 1)^3 - (2 - y_1)^3 - \frac{3}{2}(2 - y_1)^2(y_1 - 1) \right. \\
&\quad \left. + \frac{3}{4}(y_1 - 1) - (y_1 - 1)^3 + \frac{3}{4}(z_1 - y_1) \right). \\
&\geq \frac{1}{6} \left(1 + 2(x_1 - 1)^3 + \frac{3}{4}(z_1 - 1) + (z_1 - 1)^3 - (y_1 - 1)^3 \right. \\
&\quad \left. - (2 - y_1)^3 - \frac{3}{2}(2 - y_1)^2(y_1 - 1) \right) \\
&= \frac{1}{6} \left(2(x_1 - 1)^3 + \frac{3}{4}(z_1 - 1) + (z_1 - 1)^3 + \frac{3}{2}(y_1 - 1) - \frac{3}{2}(y_1 - 1)^3 \right) \\
&> \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9}\sqrt{\frac{1}{10}} \right). \tag{4.61}
\end{aligned}$$

As a conclusion of (4.59), (4.60) and (4.61), in this subcase we have

$$v(D'_1(C, X')) > \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9}\sqrt{\frac{1}{10}} \right).$$

As a conclusion of Subcases 2.1–2.10, we get

$$v(D'_1(C, X')) > \frac{1}{6} \left(\frac{5}{9} - \frac{4}{9}\sqrt{\frac{1}{10}} \right).$$

Thus, together with Case 1, Lemma 4.5 is proved. \square

Clearly, Theorem 4.1 follows from Lemmas 4.1–4.5.

As a consequence of Lemma 2.1, Lemma 3.2, Theorem 4.1 and (1.3) we get the following upper bounds for $\delta^t(C)$ and $\delta^t(T)$.

Theorem 4.2.

$$\delta^t(C) \leq \frac{90\sqrt{10}}{95\sqrt{10} - 4} \approx 0.9601527 \dots \quad \text{and} \quad \delta^t(T) \leq \frac{36\sqrt{10}}{95\sqrt{10} - 4} \approx 0.3840610 \dots.$$

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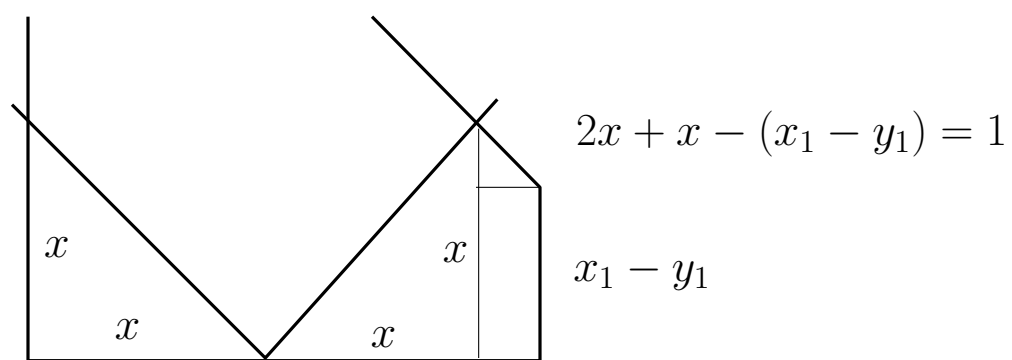


Figure 4